Nonlinear cointegration: Theory and Application to Purchasing Power Parity

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Abstract

In this paper, we study a smooth-transition type of nonlinear cointegration among a dynamic system, in which the proposed definition nests Engle and Granger (1987)'s linear cointegration. Based on the Smooth Transition Autoregressive (STAR) models, a triangular representation for the nonlinearly cointegrated system is introduced. Furthermore, two tests for nonlinear cointegration are derived in this paper: one is a residual-based test for testing the null hypothesis of no nonlinear cointegration and the other is a test for testing the hypothesis of linear cointegration against nonlinear cointegration. The statistical properties of the second test are investigated. An empirical example is illustrated by applying the first testing procedure to the dollar/lira real exchange rate. It is found that there is no linear cointegration but a nonlinearly cointegrating relation in the investigated purchasing power parity (PPP). The PPP puzzle in Hamilton (1994) can be solved by applying our nonlinear cointegration.

Key words: Dickey-Fuller test, Logistic smooth transition, Nonlinear cointegration

1. Introduction

Since introduced in Engle and Granger (EG) (1987)'s creative articles, cointegration has become an influential industry in modeling macroeconomic time series in econometrics. The EG's linear cointegration has been well developed in linear models for modeling long-run relationship among economic variables and has been widely applied in empirical studies. However, by numerical empirical findings, many economic time series, such as unemployment rates, GDP growth, interest rates, always exhibit nonlinearities. For a survey of nonlinear dynamic models, see van Dijk, Teräsvirta and Franses (2002).

On the other hand, economic relationships often display nonlinearities such as structural change and regime-switching. Recently, many researchers have already attempted to extend EG's cointegration to a nonlinear framework. For instance, Johansen's (2006) research allows for nonlinear short-run dynamics in error correction models. Balke and Fomby (1997) introduce a threshold cointegration model that accommodates for nonlinear adjustments towards a long-run equilibrium and Enders and Falk (1998) employ a similar model to investigate the purchasing power parity.

In this paper, we propose a new nonlinear cointegration, which is a joint analysis of nonlinearity and cointegration. For a nonlinear consideration we adopt one of the most popular nonlinear dynamic models, the smooth transition autoregressive (STAR) model, which nests a linear autoregressive model and contains a regime-switching structure. For a detailed discussion of STAR models, see Teräsvirta (1994). Furthermore, a constant cointegrating vector defined in Engle and Granger (1987) could be interpreted as a special case if a threshold type of cointegration is evident in the macroeconomic time series system. Therefore, the cointegrating vector can be specified in a form of a smooth transition function if there is a regime-switching in the system.

Empirical support for the theory of purchasing power parity (PPP) has been relatively mixed. The null hypothesis of no linear cointegration of EG's (1987) can not be rejected at 5% significant level when applying a residual-based testing procedure to the dollar/lira real exchange rate. See Hamilton (1994) for a detailed discussion. The PPP puzzle can be solved by introducing a nonlinear cointegration concept and the empirical results show that the economic hypothesis that the PPP system is cointegrated is accepted.

The paper is organized as follows. In Section 2 we interpret our nonlinear cointegration by an example of a bivariate system that is nonlinearly cointegrated. Section 3 gives the general definition of nonlinear cointegration. Section 4 presents the examination of the long-run equilibrium relation among nonstationary variables, for specific nonlinearities within cointegrating vector and propose a procedure of testing linear cointegration against nonlinear cointegreation. We recall the empirical example of purchasing power parity and apply our testing nonlinear cointegration procedure in Section 5. The last section is the conclusion.

2. An example of nonlinear cointegration

In this section, we illustrate a bivariate autoregressive process that each individual process is I(1) and their nonlinear combination is weakly stationary as follows:

Let $\mathrm{y}_t = (y_{1t}, y_{2t})'$ be a bivariate process defined as

$$\begin{cases} y_{1t} = \alpha_{2t} y_{2t} + \varepsilon_{1t} \\ y_{2t} = y_{2,t-1} + \varepsilon_{2t} \end{cases} \qquad (1)$$

where $\varepsilon_{it} \sim nid(0,1)$, $i = 1,2$, $E(\varepsilon_{1t}\varepsilon_{2t}) = 0$

$$\alpha_{2t} = 1 + G_{2t}$$

$G_{2t} = (1 + \exp(-\Delta y_{2t}))^{-1}$ is a logistic smooth transition function

Figure 1 plots realizations of y_{1t} and y_{2t} individually for G_{2t} and independent $N(0,1)$ variables ε_{1t} and ε_{2t}

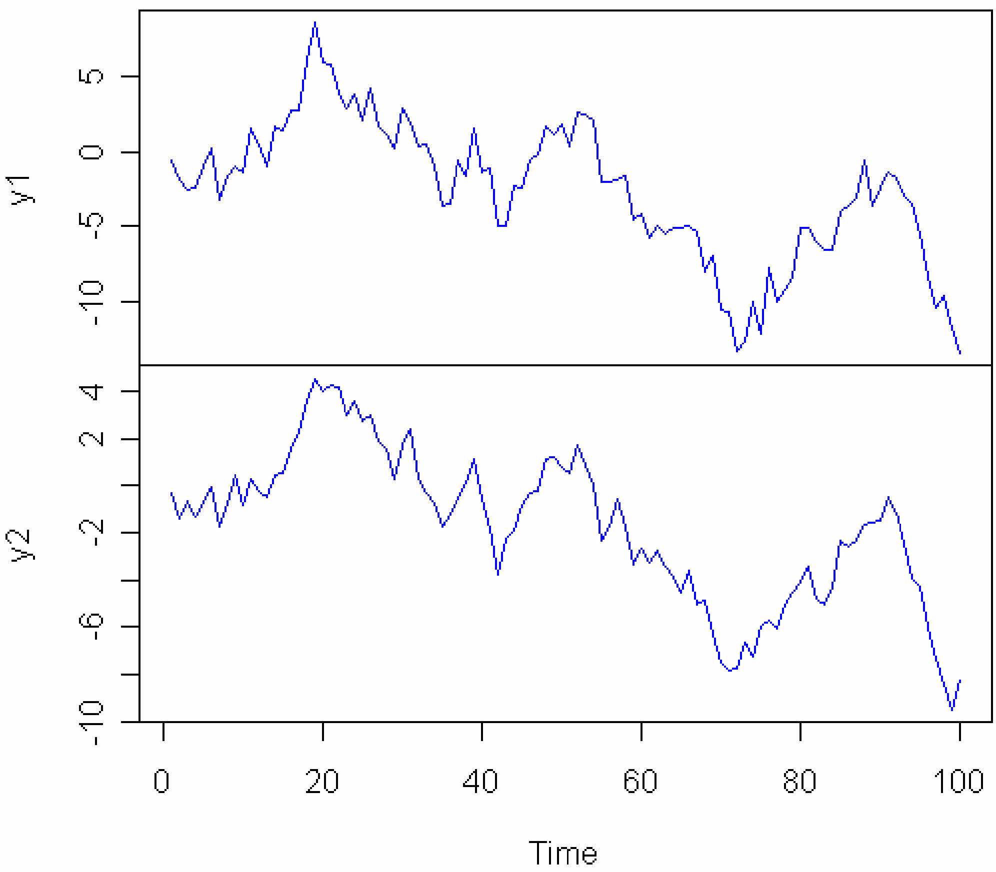

Figure 1: Realizations of y_{1t} and y_{2t} generated from equation (1)

Obviously, y_{2t} is a linear I(1), since it is a pure random walk in equation (1). If y_{1t} tends to be an I(1) process, we would conclude that there exists a cointegration in this system where the cointegrating vector is time varying rather than simple constants.

From Figure 1, we observe that there might be time trend in y_{1t}. We apply Dickey-Fuller test to prove that whether y_{1t} is nonstationary. We test

$H_0 : y_{1t} = y_{1,t-1} + u_t$ against $H_1 : y_{1t} = \rho y_{1,t-1} + u_t$, where $u_t \sim iid\left(0, \sigma^2\right)$. Hence the null hypothesis is $\rho = 1$ and the alternative hypothesis is $\rho < 1$.

The statistic of Dickey-Fuller ρ test is defined as $T\left(\hat{\rho}_T - 1\right)$; the statistic of Dickey-Fuller t test is defined as $\frac{\hat{\rho}_T - 1}{\hat{\sigma}_{\hat{\rho}_T}}$, where T is the sample size, $\hat{\rho}_T$ is the estimate of the coefficient ρ and $\hat{\sigma}_{\hat{\rho}_T}$ is the standard error of $\hat{\rho}_T$ in the estimated model $y_{1t} = \rho y_{1,t-1} + u_t$.

The estimation result of this example is $y_{1t} = \underset{(0.0358)}{0.9714} y_{1,t-1}$

In this example, $T = 100$ and the value of the test statistics are

$$T\left(\hat{\rho}_T - 1\right) = 100\left(0.9714 - 1\right) = -2.86$$

$$t = \frac{\hat{\rho}_T - 1}{\hat{\sigma}_{\hat{\rho}_T}} = \frac{0.9714 - 1}{0.0358} = -0.80$$

The asymptotic distributions of both statistics are constructed under the unity assumption of ρ. Comparing the value of each statistic at a 5% significant level of Table B.5 or Table B.6 in Hamilton (1994) (T=100), we find that neither the values of the above test statistics falls into the reject region. There is not sufficient evidence to reject the null hypothesis, we conclude that y_{1t} is an I(1) process.

Then we plot the realizations of y_{1t} and y_{2t} together in Figure 2 and inspect the relationship between these two time series.

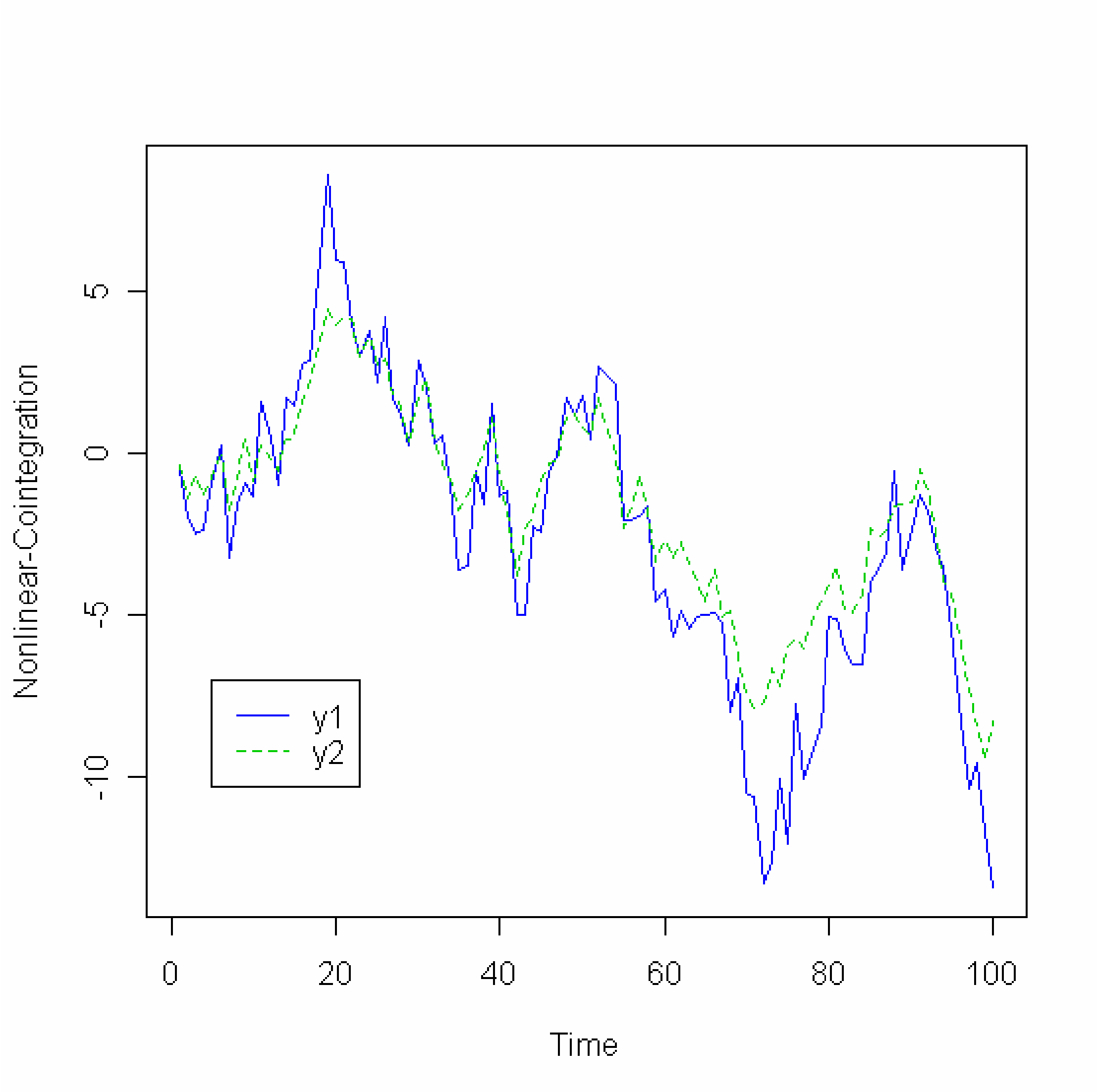

Figure 2: Comparison of realization of y_{1t} and y_{2t}

From Figure 2, we note that either series $\{y_{1t}, y_{2t}\}$ will fluctuate randomly far from the starting value. They move simultaneously but not identically. It can be seen that y_{1t} should remain some relationship with y_{2t}, which comes from the providing nonlinear regression coefficient α_{2t}. In other words, a time-varying vector as $\alpha_t = (1, -\alpha_{2t})'$ leads the nonlinear combination of two time series to be a stationary process.

From the above example, we conclude that there exists a cointegration with a time-varying vector rather than simple constants.

3. Definition of nonlinear cointegration

Engle and Granger (1987)'s definition of cointegration refers to linear combination of nonstationary variables and it describes a long-run equilibrium relation existing in a linear dynamic system. In this section, we shall generalize such linear cointegreation to a nonlinear cointegration based on a nonlinear dynamic system.

Definition Let the n-dimensional vector y_t be an I(1) process, that is for any $i = 1,\cdots,n$, $y_{it} \sim \mathrm{I}(1)$. Here, the I(1) may follow from Engle and Granger (1987)'s definition.

$\{y_{it}\}$ is said to be nonlinearly cointegrated if there exists a time-varying vector $\alpha_t = (\alpha_{1t}, \alpha_{2t} \cdots \alpha_{nt})'$ which satisfy

- The first item α_{1t} is a nonzero-constant that a normalized $\alpha_t = (1, \alpha_{2t} \cdots \alpha_{nt})'$;
- $\{\alpha_{it}\}, i = 2,\cdots,n$ are well-defined function of a random variable S_t such that for each *i*, α_{it} has a logistic smooth transition function form:

$$\alpha_{it} = \alpha_i + G_{it}$$

  where $G_{it} = (1 + \exp\{-\gamma_i (S_t - c_i)\})^{-1}$, α_i, γ_i and c_i are parameters, $\gamma_i \geq 0$

- $\alpha_t' \mathrm{y}_t \sim \mathrm{I}(0)$, i.e., a nonlinear combination of y_t is I(1).

Here, α_t is called nonlinear cointegrating vector.

Figure 3 plots a realization of G_{it} for some i and we can see that this series varies along the time trend between 0 and 1.

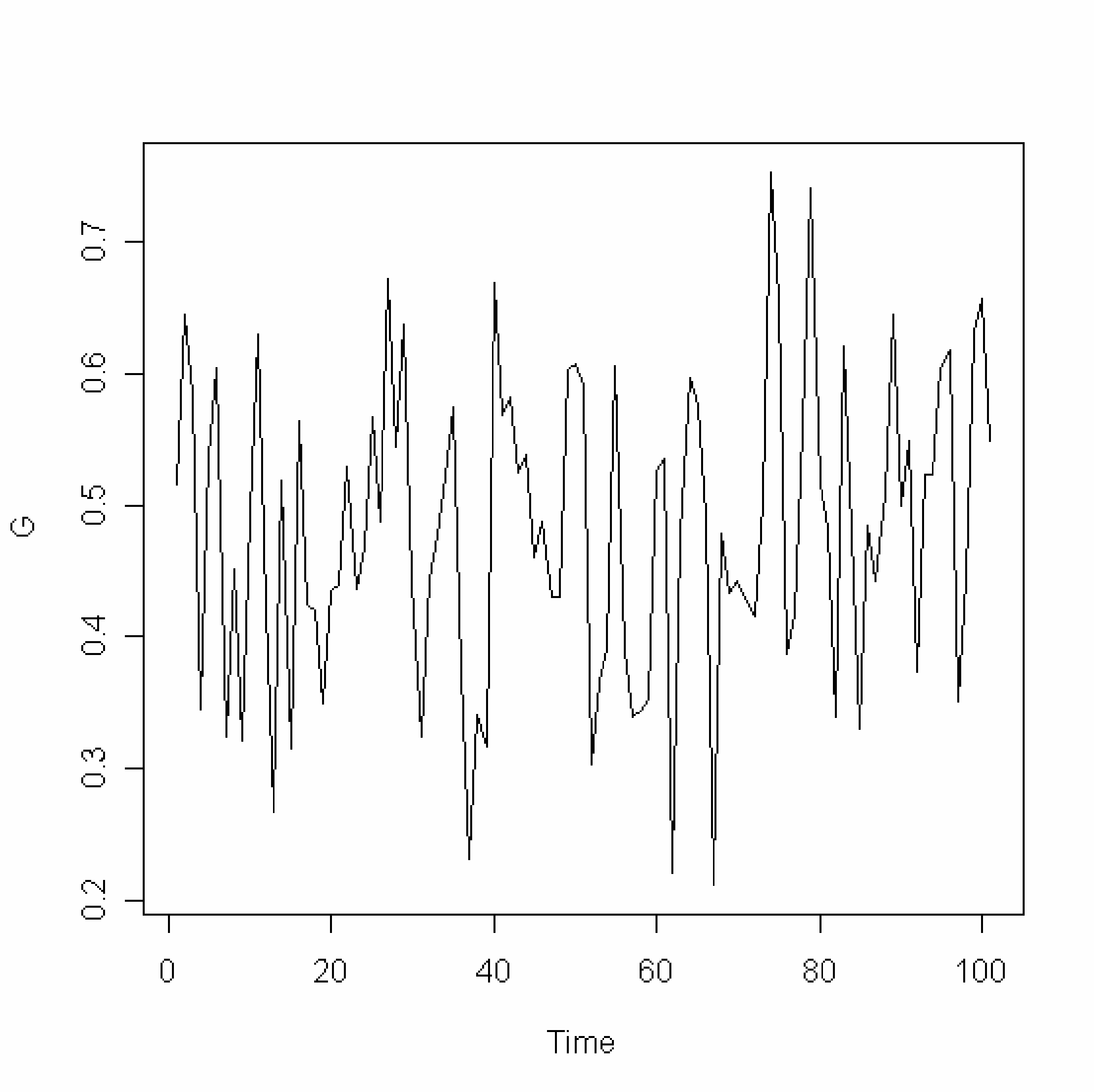

Figure 3: Plot of a realization of G_{it} where $\gamma_i = 0.5$ and $c_i = 0.1$

The logistic smooth transition function is utilized due to viewing G_{it} as a function of S_t, with γ_i and c_i fixed, it is bounded as $0 \le G_{it} \le 1$ and viewing G_{it} as a function of γ_i, with S_t and c_i fixed, the limits of G_{it} as $\gamma_i \to 0$ and $\gamma_i \to \infty$ are constants.

In this paper, we assume $S_t = \Delta y_t$ in later discussion.

Notes:

- The above definition of nonlinear cointegration nests Engle and Granger (1987)'s definition if setting $G_{it} = 0$, for all $i = 2, \cdots, n$
- It is possible to test, under the null hypothesis of $\gamma_i = 0$, linear cointegration against nonlinear cointegration among a dynamic system $\{y_{it}\}$.

4. Testing nonlinear cointegration

4.1 Test no nonlinear cointegration against nonlinear cointegration

In some economic relationships, no linear cointegration has been tested so we are strongly curious about whether any nonlinearly cointegrating relation exists. In this subsection, we assume the cointegrating vector α_t is known we assume further $\{y_{it}\}$ has a triangular representation formulated as:

$$y_{1t} = \alpha_{2t} y_{2t} + \cdots + \alpha_{nt} y_{nt} + z_t \qquad (3)$$

where $y_{it} \sim \mathrm{I}(1)$ for each $i = 1, 2, \cdots, n$

In the system (3), we wish to test no nonlinear cointegration against nonlinear cointegration and we apply a residual-based test. The null hypothesis is that $z_t \sim \mathrm{I}(1)$ whereas the alternative is $z_t \sim \mathrm{I}(0)$. Thus, as the nonlinear cointegrating vector is given, the classical Dickey-Fuller tests apply.

To carry out this test, we run an autoregression of z_t and derive the Dickey-Fuller ρ test and Dickey-Fuller *t* test after calculating the series of residuals z_t by using the known vector directly.

There are different cases in testing unit root with or without drift or time trend, see Hamilton (1994). The common used hypotheses are

$$case1:\ H_0: z_t = z_{t-1} + u_t \qquad H_0: \rho = 1$$
$$\Leftrightarrow$$
$$H_1: z_t = \rho z_{t-1} + u_t \qquad H_1: \rho < 1$$

$$case2:\ H_0: z_t = z_{t-1} + u_t \qquad H_0: \alpha = 0,\ \rho = 1$$
$$\Leftrightarrow$$
$$H_1: z_t = \alpha + \rho z_{t-1} + u_t \qquad H_1: \alpha \neq 0,\ \rho < 1$$

$$case4:\ H_0: z_t = \alpha + z_{t-1} + u_t \qquad H_0: \rho = 1,\ \delta = 0$$
$$\Leftrightarrow$$
$$H_1: z_t = \alpha + \rho z_{t-1} + \delta t + u_t \qquad H_1: \rho < 1,\ \delta \neq 0$$

where $u_t \sim iid(0, \sigma^2)$

The test statistic of Dickey-Fuller ρ test is defined as $T(\hat{\rho}_T - 1)$ for the above three cases, where T stands for the sample size and $\hat{\rho}_T$ stands for the estimate of the coefficient ρ .The test statistic of Dickey-Fuller t test is defined as $t = \dfrac{\hat{\rho}_T - 1}{\hat{\sigma}_{\hat{\rho}_T}}$, where $\hat{\sigma}_{\hat{\rho}_T}$ represents the estimated standard error of $\hat{\rho}_T$.

If the null hypothesis of $z_t \sim \mathrm{I}(1)$ is rejected, we conclude that there exists a nonlinear cointegration in the system, otherwise there is no nonlinear cointegration in the system.

The asymptotic distribution of Dickey-Fuller ρ test and t test is summarized in the Table B.5 and Table B.6 in Hamilton (1994). Check the critical value under the special case and the sample size and compare with the obtained results of the test statistics.

The smaller values of test statistics indicate that the coefficient $\rho < 1$ is significant and we comment that the nonlinear combination of I(1) processes is I(0) which implies the existence of nonlinear cointegration.

4.2 Test linear cointegration against nonlinear cointegration

Suppose we prove that the cointegration is linear, it is easy to follow the discussion in Hamilton (1994) and Johansen (2006) for either known or unknown cointegrating vector cases. In this subsection, we trace the procedure of generating test and try to recognize which type of cointegration is, linear or nonlinear.

Let the system $\{y_{it}\}$ be nonlinearly cointegrated such that y_{1t} has a nonlinear cointegration with y_{it} where $i = 1,2,\cdots,n$ but $\{y_{it}\}$, $i = 1,2,\cdots,n$ is a subsystem without any cointegration. The triangular representation of $\{y_{it}\}$ is:

$$y_{1t} = \alpha_{2t} y_{2t} + \cdots + \alpha_{nt} y_{nt} + z_t \qquad (4)$$

where $y_{it} \sim \mathrm{I}(1)$ for each $i = 1,2,\cdots,n$, $z_t \sim \mathrm{I}(0)$

for each $i = 2,\cdots,n$, α_{it} has a logistic smooth transition function form defined in Section 3.

When $\gamma_i \to 0$, α_{it} are reduced to constant terms. Thus the system (4) could be used to testing linear cointegration against nonlinear cointegration. For a convenient discussion we consider a bivariate system as follows:

$$\begin{cases} y_{1t} = \alpha_{2t} y_{2t} + u_t \\ y_{2t} = y_{2,t-1} + v_t \end{cases} \qquad (5)$$

where the error terms of u_t and v_t satisfy:

$u_t \sim iid(0,\sigma_u^2)$ with $\sigma_u^2 > 0$ and $Eu_t^4 < \infty$; $v_t \sim iid(0,\sigma_v^2)$ with $\sigma_v^2 > 0$ and $Ev_t^4 < \infty$ and $Eu_t v_\tau = 0$ for any t and τ ;

$$\alpha_{2t} = \beta + G_t(\Delta y_{2t};\gamma,c) \quad , \quad G_t(\Delta y_{2t};\gamma,c) = \frac{1}{1+\exp\{-\gamma(\Delta y_{2t}-c)\}} - \frac{1}{2}, \gamma \geq 0 .$$

The first equation in model (5) is expressed as

$$y_{1t} = \beta\, y_{2t} + G_t(\Delta y_{2t};\gamma,c) y_{2t} + u_t \qquad (6)$$

In equation (6), $\beta\, y_{2t}$ stands for a linear cointegration part and $G_t(\Delta y_{2t};\gamma,c) y_{2t}$ stands for a nonlinear cointegration part. The feature that the limit of $G_t(\Delta y_{2t};\gamma,c)$ is zero, as γ going to zero reduces the nonlinear regression to linear regression.

Hence, the aim of testing whether the nonlinear part is essential in the cointegrating procedure is interpreted in the following hypotheses:

$$\begin{array}{lcl} H_0 : y_{1t} = \beta\, y_{2t} + u_t & & H_0 : \gamma = 0 \\ & \Leftrightarrow & \\ H_1 : y_{1t} = \beta y_{2t} + G_t(\Delta y_{2t};\gamma,c) y_{2t} + u_t & & H_1 : \gamma > 0 \end{array}$$

In order to avoid the identification problem which arises from the setting of $\gamma = 0$, we adopt a first-order Taylor expansion of γ around 0 in the transition function $G_t(\Delta y_{2t};\gamma,c)$, introduced by He and Sandberg (2006).

The result of the first-order Taylor approximation is

$$G_t(\Delta y_{2t};\gamma,c) = \frac{\gamma(\Delta y_{2t}-c)}{4} + R(\gamma) \qquad (7)$$

where $R(\gamma)$ is a remainder term which depends on γ and converges to zero as γ

going to zero.

We substitute equation (7) into the model in equation (6), after merging terms, we obtain

$$y_{1t} = \left(\beta - \frac{\gamma c}{4} \right) y_{2t} + \frac{\gamma}{4} \Delta y_{2t} y_{2t} + R(\gamma) y_{2t} + u_t \qquad (8)$$

We rewrite equation (8) as $$y_{1t} = \beta^* y_{2t} + \phi \Delta y_{2t} y_{2t} + u_t^* \qquad (9)$$

where $\beta^* = \beta - \frac{\gamma c}{4}$, $\phi = \frac{\gamma}{4}$, $u_t^* = R(\gamma) y_{2t} + u_t$

Therefore, the null hypothesis of $\gamma = 0$ is identical to

$$H_0 : \beta^* = \beta \, , \phi = 0 \, , u_t^* = u_t \qquad (10)$$

Then we turn to consider the auxiliary regression

$$y_{1t} = \beta^* y_{2t} + \phi \Delta y_{2t} y_{2t} + u_t^* \qquad (11)$$

The asymptotic distribution can be derived by Functional Central Limit Theorem,

$$T^{-2} \sum y_{2t}{}^2 \xrightarrow{d} \sigma_v^2 \int_0^1 [W(r)]^2 dr$$

$$T^{-2} \sum \Delta y_{2t}{}^2 y_{2t}{}^2 \xrightarrow{d} \sigma_v^4 \int_0^1 [W(r)]^2 dr$$

$$T^{-2} \sum \Delta y_{2t} y_{2t}{}^2 \xrightarrow{p} 0$$

$$T^{-1} \sum y_{2t} u_t \xrightarrow{d} \frac{1}{2} \sigma_u \sigma_v \left\{ [W(1)]^2 - 1 \right\}$$

$$T^{-1} \sum \Delta y_{2t} y_{2t} u_t \xrightarrow{d} \frac{1}{2} \sigma_u \sigma_v^2 \left\{ [W(1)]^2 - 1 \right\}$$

Under the null hypothesis in (10)

$$\begin{pmatrix} T\left(\hat{\beta}^{*}-\beta\right) \\ T\hat{\phi} \end{pmatrix} = \begin{pmatrix} T^{-2}\sum y_{2t}{}^{2} & T^{-2}\sum \Delta y_{2t} y_{2t}{}^{2} \\ T^{-2}\sum \Delta y_{2t} y_{2t}{}^{2} & T^{-2}\sum \Delta y_{2t}{}^{2} y_{2t}{}^{2} \end{pmatrix}^{-1} \begin{pmatrix} T^{-1}\sum y_{2t} u_{t} \\ T^{-1}\sum \Delta y_{2t} y_{2t} u_{t} \end{pmatrix}$$

$$\xrightarrow{d} \begin{pmatrix} \sigma_v^2 \int_0^1 [W(r)]^2 dr & 0 \\ 0 & \sigma_v^4 \int_0^1 [W(r)]^2 dr \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{2}\sigma_u \sigma_v \left\{[W(1)]^2 - 1\right\} \\ \frac{1}{2}\sigma_u \sigma_v^2 \left\{[W(1)]^2 - 1\right\} \end{pmatrix}$$

Thus,

$$T\left(\hat{\beta}^{*}-\beta\right) \xrightarrow{d} \left[\sigma_v^2 \int_0^1 [W(r)]^2 dr\right]^{-1} \left[\frac{1}{2}\sigma_u \sigma_v \left\{[W(1)]^2 - 1\right\}\right] = \frac{\sigma_u \left\{[W(1)]^2 - 1\right\}}{2\sigma_v \int_0^1 [W(r)]^2 dr} \qquad (12)$$

$$T\hat{\phi} \xrightarrow{d} \left[\sigma_v^4 \int_0^1 [W(r)]^2 dr\right]^{-1} \left[\frac{1}{2}\sigma_u \sigma_v^2 \left\{[W(1)]^2 - 1\right\}\right] = \frac{\sigma_u \left\{[W(1)]^2 - 1\right\}}{2\sigma_v^2 \int_0^1 [W(r)]^2 dr} \qquad (13)$$

Besides the estimated values of $\hat{\beta}^{*}$ and $\hat{\phi}$, associated estimated error terms are determined from auxiliary regression in (11),

$$\hat{u}_t^{*} = y_{1t} - \hat{\beta}^{*} y_{2t} - \hat{\phi}\,\Delta y_{2t} y_{2t}$$

Under the null, we achieve $\hat{u}_t = \hat{u}_t^{*}$, then the estimated standard deviation of $\hat{u}_t$ is

$$\hat{\sigma}_u = \sqrt{\operatorname{var} \hat{u}_t} = \sqrt{\operatorname{var} \hat{u}_t^{*}} = \sqrt{\frac{1}{T}\sum \left(\hat{u}_t^{*} - E\left(\hat{u}_t^{*}\right)\right)^2}$$

From model in equation (5), $v_t = \Delta y_t$, then the estimated standard deviation of v_t is

$$\hat{\sigma}_v = \sqrt{\operatorname{var} v_t} = \sqrt{\operatorname{var} \Delta y_t} = \sqrt{\frac{1}{T}\sum \left(\Delta y_t - E\left(\Delta y_t\right)\right)^2}$$

Therefore, we construct the adjusted test statistics after (12) and (13) as follows:

$$T\frac{\hat{\sigma}_v}{\hat{\sigma}_u}\left(\hat{\beta}^* - \beta\right)\xrightarrow{d}\frac{\left\{[W(1)]^2 - 1\right\}}{2\int_0^1 [W(r)]^2 dr} \qquad (14)$$

$$T\frac{\hat{\sigma}_v^2}{\hat{\sigma}_u}\hat{\phi}\xrightarrow{d}\frac{\left\{[W(1)]^2 - 1\right\}}{2\int_0^1 [W(r)]^2 dr} \qquad (15)$$

It is clear that either asymptotic distribution of adjusted test statistics is equivalent to the asymptotic distribution of Dickey-Fuller ρ test of Case 1, see Hamilton (1994). The critical values of above two test statistics for different sample size are tabulated in Table 1.

Table 1: Critical Values for Adjusted Test Statistic based on OLS Estimation of Auxiliary Regression

Sample Size T	**Probability that Adjusted Test Statistic is less than entry**							
	0.01	**0.025**	**0.05**	**0.1**	**0.9**	**0.95**	**0.975**	**0.99**
25	-11.9	-9.3	-7.3	-5.3	1.01	1.40	1.79	2.28
50	-12.9	-9.9	-7.7	-5.5	0.97	1.35	1.70	2.16
100	-13.3	-10.2	-7.9	-5.6	0.95	1.31	1.65	2.09
250	-13.6	-10.3	-8.0	-5.7	0.93	1.28	1.62	2.04
500	-13.7	-10.4	-8.0	-5.7	0.93	1.28	1.61	2.04
∞	-13.8	-10.5	-8.1	-5.7	0.93	1.28	1.60	2.03

Given the interested value of β, we calculate the values of both adjusted test statistics. If either reported value is smaller than the critical value, the null hypothesis of linear cointegration is rejected. It is strongly suggestive of the existence of nonlinear cointegration.

5. Empirical analysis------Purchasing Power Parity

5.1 Theory of purchasing power parity

A classical economic example of cointegration interpretation is the purchasing power parity. The basic proposal of purchasing power parity is first developed by Gustav Cassel in 1920. It is a method based on the law of one price, in which the idea is that, in an efficient market, the identical goods must have only one price. The ratio of the prices in different currencies is the exchange rate.

The purchasing power parity exchange rate equalizes the purchasing power of different currencies in their home countries for a given basket of goods. These kind of special exchange rates are often used to compare the standards of living of the different countries. Here, using purchasing power parity exchange rates is preferred to using market exchange rates because of the significant difference between purchasing power parity exchange rate and market exchange rate. Market exchange rates fluctuate frequently due to the need of the market while many economists believe that purchasing power parity exchange rates are characterized by a long-run equilibrium.

The measurement of purchasing power is the domestic price level in each country. The price level growth means that the purchasing power of the currency of this country decreases and the currency is devalued at the same rate, and vice versa. In fact, the measurement of purchasing power parity is complicated since that there is no simply uniform price level differed among countries. It is necessary to compare the cost of baskets of goods and services consumed by people from different countries using a price index. Apparently, the consumer price index organized monthly is the best choice to assess the purchasing power.

In economics, a consumer price index is a statistical time series of a weighted average of prices of a specified set of goods and services purchased by the consumers. It is a price index that tracks the prices of a specified basket of consumer goods and services.

5.2 Data description

The fundamental idea of purchasing power parity holds that $P_t = S_t \times P_t'$, where P_t denotes the domestic price index, P_t' denotes the foreign price index and

S_t denotes the exchange rate between the currencies of two countries.

In Hamilton's book, P_t denotes a price index in United States (in dollar per good), P_t' a price index in Italy (in lira per good) and S_t the rate of exchange between the currencies of United States and Italy (in dollar per lira). If we take the logarithms of both sides of $P_t = S_t \times P_t'$, it turns to $p_t = s_t + p_t'$.

In fact, along with the transportation costs and errors in assessing prices due to different home countries, the purchasing power parity could hardly keep exactly at every time point through the time trend. Therefore, consider the econometric model $z_t = p_t - s_t - p_t'$, where z_t represents the a deviation from purchasing power parity. A weakly expression of purchasing power parity is that z_t is stationary even though the individual elements of $\{p_t, s_t, p_t'\}$ are all I(1).

For analysis objectives, the raw data are transformed at first as presented:

$$p_t = 100 \times [\log(P_t) - \log(P_{1973:1})]$$

$$p_t' = 100 \times [\log(P_t') - \log(P_{1973:1}')]$$

$$s_t = 100 \times [\log(S_t) - \log(S_{1973:1})]$$

Such transition is to make sure that above three time series have same starting values, zero at January 1973. Multiplying by 100 roughly represents the percentage difference between the value at current time point and the starting value at January 1973.

Figure 4 plots the monthly data of the price level p_t in United States, the price level p_t' in Italy and the exchange rate s_t between dollar in United States and lira in Italy from January 1973 to October 1989 after the transition.

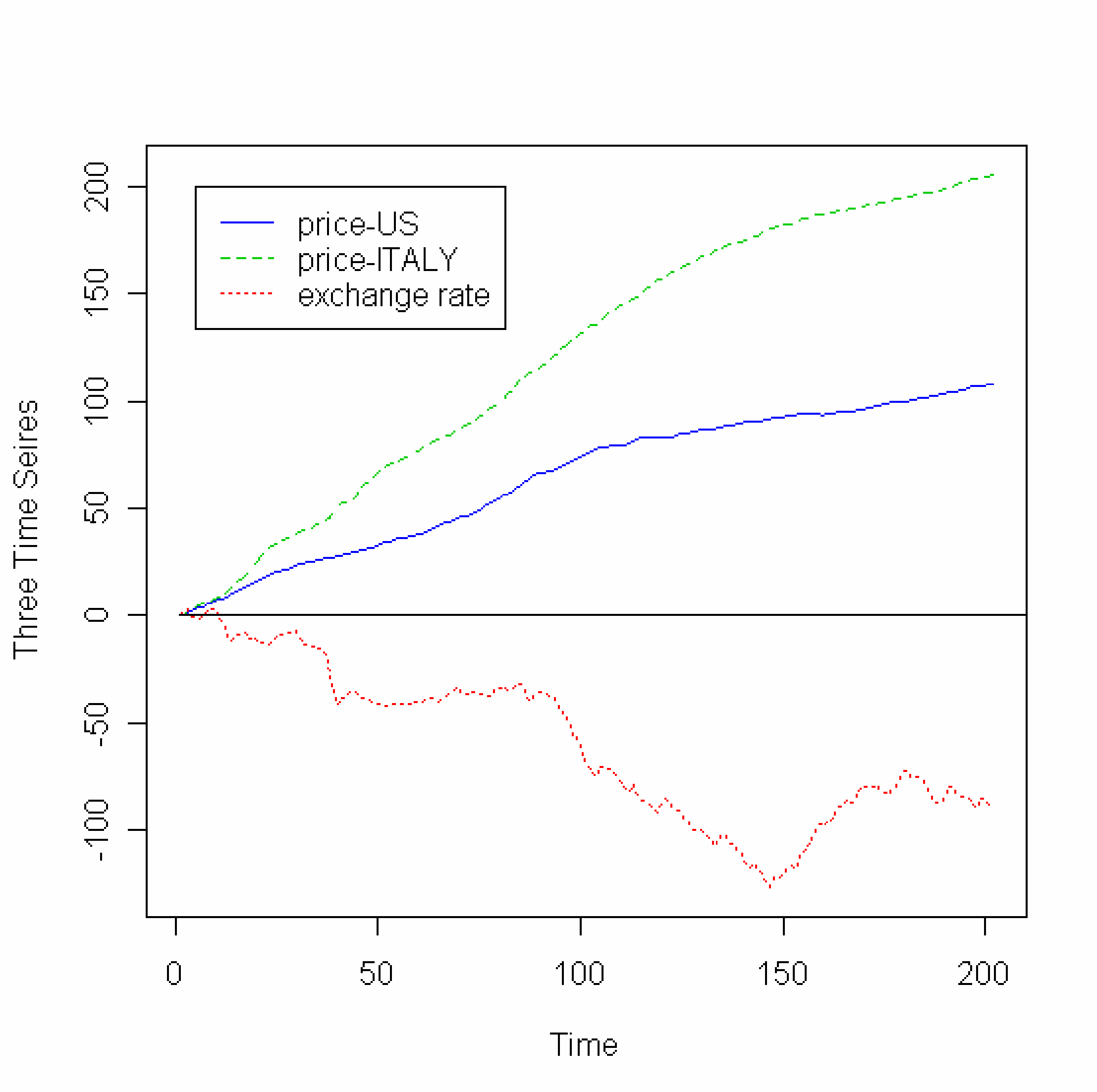

Figure 4: Monthly data of p_t, s_t and p_t' from 1973 to 1989 after transition

5.3 No Linear cointegration in PPP

We review the example of purchasing power parity discussed in Hamilton's book. In Figure 4, it appears that each time series might be a I(1) process. The problem solving process starts from testing the individual elements of $\{p_t, s_t, p_t'\}$ are all I(1) process. For the monthly data, we apply augmented Dickey-Fuller test to prove that we can not reject the null hypothesis of nonstationary. The results of the estimation and the calculation of the test statistics are stated in details in Hamilton (1994)'s.

Based on the purchasing power parity, the consideration of the relation among the consumer price indexes and the exchange rate can be induced to investigate the existence of the conintegration for a given conintegrating vector $\alpha = (1, -1, -1)'$.

Thus, the key issue of the discussion is to test the stationarity of $z_t = p_t - s_t - p_t'$. The plot of z_t is shown in Figure 5, which describes a rough equilibrium.

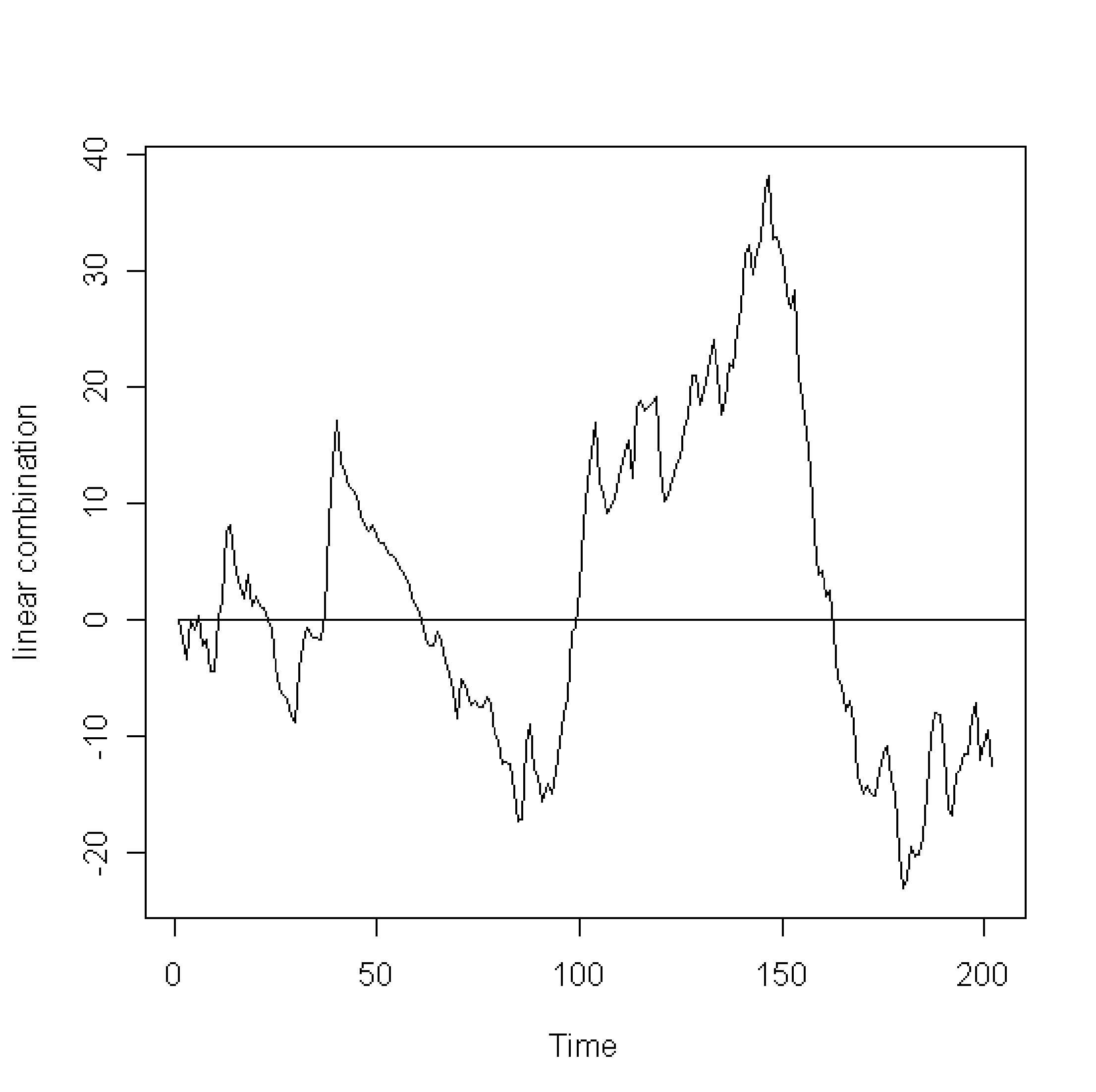

Figure 5: z_t, linear combination of p_t, s_t and p_t'

The outputs of the examination of the long-run equilibrium do not provide effective evidence to reject the hypothesis of nonstationary, hence the conintegrating vector

$\alpha = (1, -1, -1)'$ does not result in a stationary z_t .

From the research in Hamilton's, it expresses that the theory of purchasing power parity is not satisfied in the example of United States and Italy. This is widespread disagreement of the purchasing power parity.

In next section, we will bring in the nonlinear conintegration concept and introduce a particular transition function, in order to verify the applicability of the purchasing power parity by the adjustment of the conintegrating vector.

5.4 Existence of nonlinear cointegration in PPP

After studying the Hamilton (1994)'s empirical example of purchasing power parity between United States and Italy from 1973 to 1989, we apply the definition of nonlinear cointegration and go further analysis to check whether there is any cointegrating connection in the system.

According to the definition in Section 3, the interest in this section is motivated by the possibility of a specialized given nonlinear cointegrating vector, for which, we apply the test procedure stated in Section 4.1.

Suppose the particular nonlinear cointegrating vector is $\alpha_t = (1, -\alpha_{2t}, -\alpha_{3t})'$ where

$\alpha_{it} = \dfrac{1}{1+\exp\{-(\Delta y_{it})\}}$ for $i = 2, 3$ is appointed as a logistic smooth transition.

The demonstration of I(1) process for each time series of the consumer price indexes and the exchange rate has been argued in Section 5.1. The nonlinear combination of the consumer price indexes and the exchange rate is calculated by

$$z'_t = p_t - s_t \frac{1}{1+\exp\{-(\Delta s_t)\}} - p'_t \frac{1}{1+\exp\{-(\Delta p'_t)\}}$$

Graph the nonlinear combination z_t' in plot which provides a strong description of the exclusion of time trend in Figure 6.

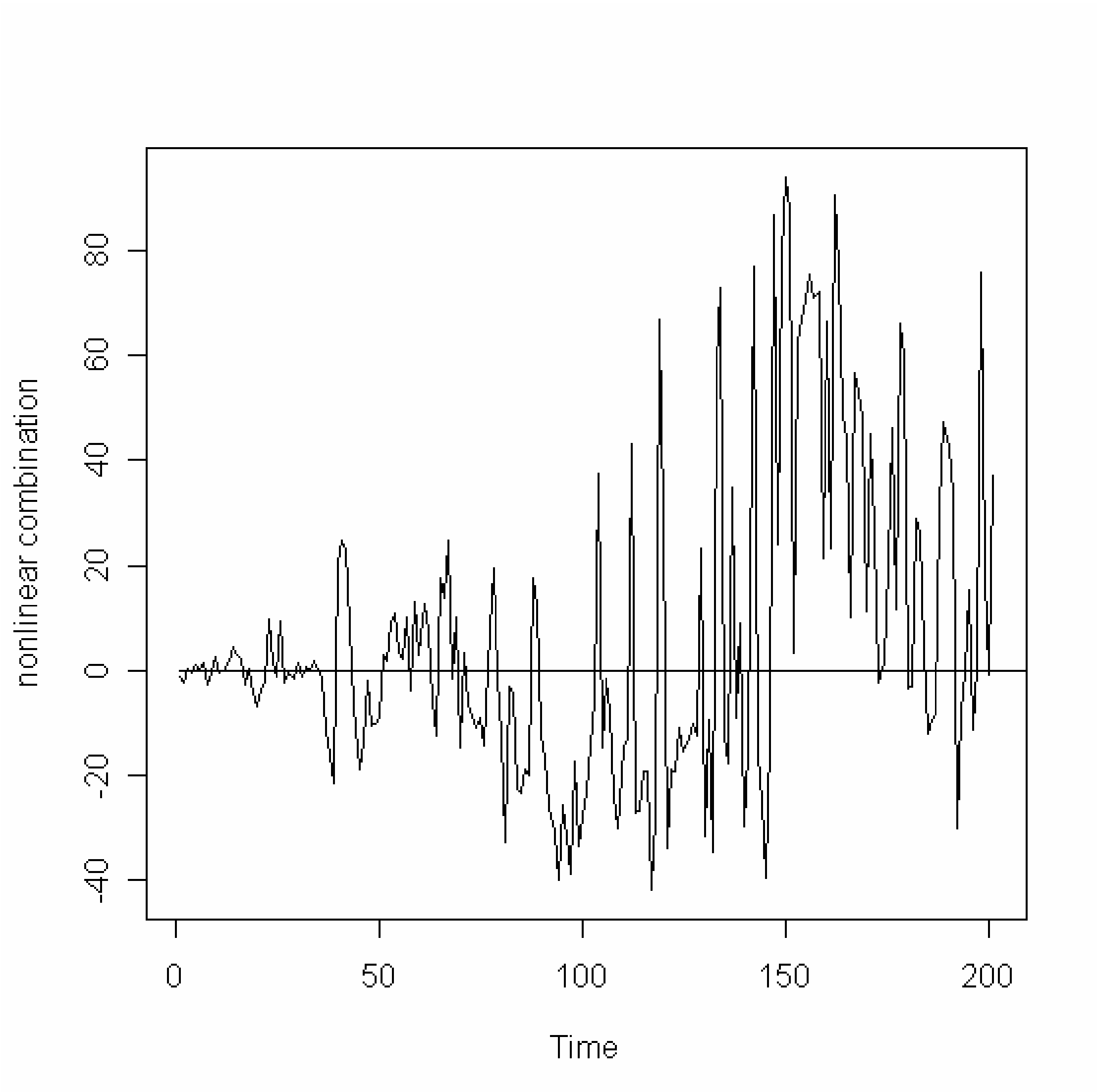

Figure 6: z_t', nonlinear combination of p_t, s_t and p_t'

Recall the knowledge in Section 4.1, we test the null hypothesis $H_0 : z_t' = z_{t-1}' + u_t$ against the alterative hypothesis $H_1 : z_t' = \alpha + \rho z_{t-1}' + u_t$, where $u_t \sim iid\left(0, \sigma^2\right)$.

The estimation result of this example is $z_t' = \underset{(1.8288)}{3.1067} + \underset{(0.0605)}{0.5299} z_{t-1}'$

In this example $T = 201$ and the value of the test statistics are

$$T(\hat{\rho}_T - 1) = 201(0.5299 - 1) = -94.49$$

$$t = \frac{\hat{\rho}_T - 1}{\hat{\sigma}_{\hat{\rho}_T}} = \frac{0.5299 - 1}{0.0605} = -7.77$$

The distributions of statistics are generated under the compounding hypothesis of the unity assumption of ρ and the zero assumption of the intercept α.

At a 5% significant level, compare the value of each statistic with the critical value from the Table B.5 or Table B.6 in Hamilton's for a sample size T=201, we find that the ρ test statistic is negative with a larger absolute value which belongs to the reject region and the value of t test statistic also provides a sufficient evidence to reject the null hypothesis, $\rho = 1$ and $\alpha = 0$

Thurs, $z_t' \sim \mathrm{I}(0)$ means the residuals of the regression p_t on s_t and p_t' is stationary, which equals to the deviation from purchasing power parity is stationary. $\{p_t, s_t, p_t'\}$ is cointegrated by $\partial_t = \left(1, -\frac{1}{1+\exp\{-(\Delta s_t)\}}, -\frac{1}{1+\exp\{-(\Delta p_t')\}}\right)'$ which is related to the difference of s_t and p_t'.

The results suggest that although all consumer indexes and exchange rate exhibit a unit root process, over a long-run, the nonlinear cointegrating vector tie the separate I(1) processes together, and tends to display a long run equilibrium attribute. Hence, purchasing power parity is warranted in the US-dollar and Italy-lira investigation.

6. Conclusion

In this paper, we introduce a smooth-transition type of nonlinear cointegration with the cointegrating vector having a dynamic structure, so that Engle and Granger (1987)'s linear cointegration is a special case here. Based on the Smooth Transition

Autoregressive (STAR) models, a triangular representation for the nonlinear cointegrating system is used, and such a dynamic structure is flexible since a linear cointegrating regression is nested in the system. In order to avoid spurious regressions, two tests of testing for nonlinear cointegration are created. The residual-based test for testing the null hypothesis of no nonlinear cointegration is easy to carry out if the economic cointegrating vector is known. The second test is based on the STAR model that containing a single nonlinear cointegration and it is used to test the linear cointegration under the null hypothesis and alternatively the model is a nonlinearly cointegrating regression. Asymptotic distribution of this test for a bivariate system is derived and it is showed that the asymptotic distribution for the adjusted test statistic is identical to the distribution of classical Dickey-Fuller ρ test under Case 1.

An empirical example is considered by applying our nonlinear cointegration theory to the dollar/lira real exchange rate. The illustration of this example shows that there is no linear cointegrating relation in the PPP system, however, there is our nonlinear cointegration evidence supported by our empirical analysis for the PPP example. It is suggested that the PPP system may contain nonlinear features between the economic variables which could cause the PPP puzzle in Hamilton (1994).

References

Balke, N. and T. B. Fomby (1997), Threshold Cointegration. *International economic Review* **38**, 627-645.

Enders W. and B. Falk (1998), Threshold-autoregressive, median unbiased, and cointegration tests of purchasing power parity. *International Journal of Forecasting* **14**, 171-186.

Engle, R. F. and C. W. J. Granger (1987), Co-integration and error-correction: Representation, estimation and testing. *Econometrica* **55**: 251-276

Hamilton, J. D. (1994), Time series analysis. Princeton University Press

He, C. and R. Sandberg (2006), Dickey-Fuller type of tests against nonlinear dynamic models. *Oxford bulletin of economics and statistics* **68**: 835-861

He, C., T. Teräsvirta and A. González (2007), Testing parameter constancy in vector autoregressive models against continuous change. *Econometric Reviews*

Johansen, S. (2006), Cointegration: An overview. Palgrave Handbooks of econometrics: Vol. 1 Econometric Theory. *Palgrave MacMillan*

Teräsvirta T. (1994), Specification, estimation, and evaluation of smooth transition autoregressive models. *Journal of the American Statistical Association* **89,** 208-218.

van Dijk, D., T. Teräsvirta and P. H. Franses (2002), Smooth transition autoregressive models - a survey of recent developments., *Econometric Reviews* **21**, 1-47