

Residual-based Test for Nonlinear Cointegration with Application in PPPs

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Abstract

Nested by linear cointegration first provided in Granger (1981), the definition of nonlinear cointegration is presented in this paper. Sequentially, a nonlinear cointegrated economic system is introduced. What we mainly study is testing no nonlinear cointegration against nonlinear cointegration by residual-based test, which is ready for detecting stochastic trend in nonlinear autoregression models. We construct cointegrating regression along with smooth transition components from smooth transition autoregression model. Some properties are analyzed and discussed during the estimation procedure for cointegrating regression, including description of transition variable. Autoregression of order one is considered as the model of estimated residuals for residual-based test, from which the test statistic is obtained. Critical values and asymptotic distribution of the test statistic that we request for different cointegrating regressions with different sample sizes are derived based on Monte Carlo simulation. The proposed theoretical methods and models are illustrated by an empirical example, comparing the results with linear cointegration application in Hamilton (1994). It is concluded that there exists nonlinear cointegration in our system in the final results.

KEY WORDS: Nonlinear cointegration; Common features; Regime transition; Residual-based test; Purchasing power parities.

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1. INTRODUCTION

Economic time series display many distinctive stylized facts, called features of these time series. Stochastic trend is one of them, which can be a feature shared in common if each individual time series has stochastic trend while linear or nonlinear combination of them does not have the feature. An economic system with this kind of feature in common will be considered as a cointegrated economic system. A modern theme of econometric studying of whether there exists common features or not among a group of economic time series is important for some reasons in the meaning of macroeconomics, economic interest or modeling.

Cointegration is based on the concept of integration from Engle and Granger (1987). The components of a vector time series are said to be cointegrated if all components are integrated of a certain order and the combination of all components is integrated of a reduced order. In Hamilton (1994), a cointegrated vector time series means that each of the series taken individually is nonstationary with one unit root, while there exists a linear combination of the series which is stationary for some nonzero vector as their cointegration vector.

Cointegrated processes in time series are known as a particular class of vector unit root processes by the definition of Hamilton (1994). After Engle and Granger (1987), a formal development comes almost in linear cointegration. However, in recent years, quite a few different nonlinear time series models have been suggested in the literature, for example the logistic and exponential smooth transition autoregressive (STAR) models by Teräsvirta (1994), which allow nonlinear dynamic structures with smooth regime transition, have been widely applied on economic data. The cointegrating regression in this paper, which is useful for the main part of testing procedure, will be derived from nonlinear dynamic structures with smooth regime transition that is the most important point different with linear cointegration. Many empirical studies of economic time series have shown strong evidence that many vector time series display nonlinear features. Nonlinear cointegration is more and more practical for economic system.

The aim of this essay is to test whether there exists nonlinear cointegration in economic time series system along with unknown cointegration vector. What we are interested is the Purchasing Power Parities (PPPs) data that have been analyzed for linear cointegration by Hamilton (1994). The theory of purchasing power parity in economy uses the long-term

equilibrium exchange rate of two currencies to equalize their purchasing power, whose basis is the "law of one price". In the absence of transportation and other transaction costs, competitive markets equalize the price of an identical good in two countries when the prices are expressed in the same currency.

Section 2 of this essay discusses two kinds of models: cointegrating regression and residuals model; Section 3 presents the residual-based test for investigating nonlinear cointegration under the condition of unknown cointegration vector in a vector time series system; Section 4 derives the table of critical values and the asymptotic distribution of test statistic by simulation; Section 5 introduces how to estimate the parameters in unknown cointegration vector; Section 6 considers an application with empirical economic PPPs data, along with applying the theoretical methods described in other sections; Section 7 contains concluding remarks and discussions in the end.

2. THE MODELS

2.1 Nonlinear cointegration

The main idea and aim in this essay is around nonlinear cointegration, rather than a linear one. The difference between nonlinear cointegration and linear cointegration is that the combination of all time series in the system is not a linear function of those series in nonlinear one, compared with a linear function in linear one. In nonlinear situation, at least one of the components in cointegration vector depends on time. The background of all sections in this essay, whatever the content is about the models, the test or the estimation, is under the definition of nonlinear cointegration which is given as follows.

Definition 1. Let $\mathbf{y}_t = (y_{1t}, y_{2t}, \dots, y_{nt})'$ denote an $(n \times 1)$ vector time series where each of the series taken individually is integrated of order 1, which means

$$y_{it} \sim I(1), \text{ for each } i = 1, 2, \dots, n. \quad (2.1)$$

The vector time series \mathbf{y}_t is said to be nonlinearly cointegrated if there exists a nonzero $(n \times 1)$ time-varying or random vector $\boldsymbol{\alpha}_t = (a_{1t}, a_{2t}, \dots, a_{nt})'$ such that

$$\boldsymbol{\alpha}'_t \mathbf{y}_t \sim I(0), \quad (2.2)$$

that is to say that the nonlinear combination of the series \mathbf{y}_t is stationary.

Under the definition of nonlinear cointegration, there should be at least one of a_{it} to be not a constant that is not correlative with time. Because the cointegration will be linear one if all components in cointegration vector are constant.

2.2 Cointegrating regression

The nonlinear cointegration vector α_t will be normalized without loss of generality for cointegrating regression. Because we denote the combination of y_t as the error term in the model of cointegrating regression for next analysis by residual estimation, while one of y_t should be considered as the regression in the model, at the same time, the others are considered as the regressors. The model, called cointegrating regression, is derived from the combination of all series y_t . The coefficient of one of them, which should not be equal to zero, is selected to be normalized to be one with the series considered as the regression in the model.

If we consider the cointegrating regression as

$$y_{1t} = a_{2t}y_{2t} + a_{3t}y_{3t} + \dots + a_{nt}y_{nt} + u_t, \quad (2.3)$$

the cointegration vector will be $\alpha_t = (1 \quad -a_{2t} \quad \dots \quad -a_{nt})'$, which have been normalized for investigating u_t for testing nonlinear cointegration. For $i > 1$, the important coefficients a_{it} in our model, whose formula is a function of y_t transformed from sigmoid function as a special case of logistic function, are considered as

$$a_{it} = \left(a_{i0} + \frac{a_i}{1 + \exp\{-\gamma_i z_{it}\}} \right), \quad \gamma_i > 0 \quad (2.4)$$

a kind of nonlinear dynamic structures with smooth regime transition that is similar with Logistic STAR model by Teräsvirta (1994) for nonlinear time series analysis.

From (2.4), if γ_i is equal to zero, a_{it} will be a constant, $(a_{i0} + 0.5a_i)$. It is said that if γ_i stays away from zero, the component a_{it} will be nonlinear dynamic structure, however, if it is near to zero, a_{it} will be linear structure.

To test nonlinear cointegration, cointegrating regression (2.3) that is considered for the estimation of residuals, \hat{u}_t , should be the model with different transition variable z_{it} in (2.4). The transition variable z_{it} is denoted as a function of y_t which may be one or more than one regimes of function of y_{it} . In this paper, four different forms of transition variables are considered as follows:

- $(\Delta y_{i,t-1} - c_i)$
- $(y_{i,t-1} - c_i)$
- $(\Delta y_{i,t-1} - c_1)(\Delta y_{i,t-1} - c_2)$
- $(\Delta y_{i,t-1} - c_1)(\Delta y_{j,t-1} - c_2), i \neq j$.

For a given set $\{a_{i0}, a_i, z_{it}\}$, for example $\{2, 0.5, (\Delta y_{i,t-1} - 1)\}$ where $\Delta y_{i,t-1}$ has standard normal distribution, $a_{it} = 2 + 0.5(1 + \exp\{-\gamma_i(\Delta y_{i,t-1} - 1)\})^{-1}$ under different γ_i are showed in Figure 1.

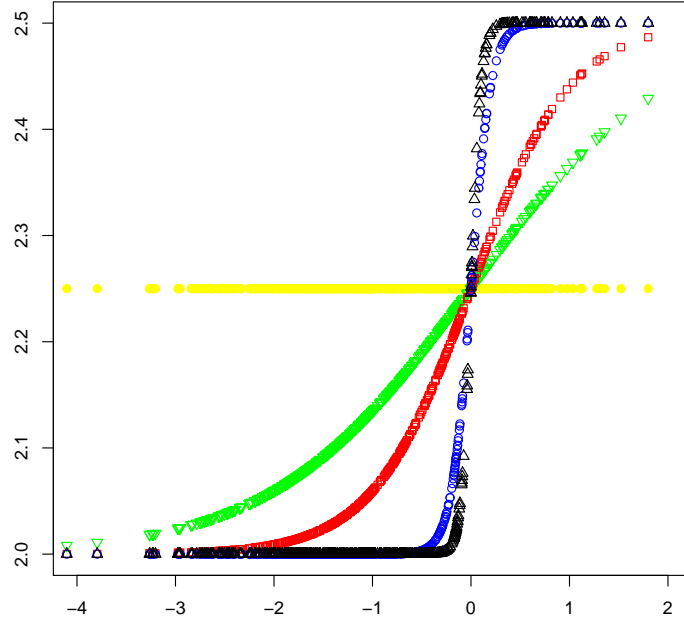


Figure 1: a_{it} as a function of $(\Delta y_{i,t-1} - 1)$ under different smooth transition components: $\gamma_i = 0$ (solid circle, yellow); $\gamma_i = 1$ (triangle point-down, green); $\gamma_i = 2$ (square, red); $\gamma_i = 10$ (circle, blue); $\gamma_i = 20$ (triangle point-up, black).

Usually, according to the speciality of true data, cointegrating regression can be adjusted from (2.3) to be

$$y_{1t} = a + a_{2t}y_{2t} + \dots + a_{nt}y_{nt} + u_t \quad (2.5)$$

including constant term, or

$$\begin{aligned}
y_{1t} = & a_{2t}y_{2t} + \dots + a_{nt}y_{nt} \\
& + \beta_{21}\Delta y_{2,t-1} + \dots + \beta_{2p}\Delta y_{2,t-p} \\
& + \dots \\
& + \beta_{n1}\Delta y_{n,t-1} + \dots + \beta_{np}\Delta y_{n,t-p} \\
& + u_t
\end{aligned} \tag{2.6}$$

adding linear components for finding a better fitness. Actually, there is divided into two parts in (2.3): linear one described by a_{i0} and nonlinear one by the others. The linear components that is added in (2.6) raise another question how to choose an appropriate value of delay parameter p for model selection, which is an application issue depends on data.

2.3 An example

We take a specific case for example under the definition of nonlinear cointegration with three time series $y_t = (y_{1t}, y_{2t}, y_{3t})'$ in the system:

$$\begin{cases} y_{1t} = a_{2t}y_{2t} + a_{3t}y_{3t} + u_t \\ y_{2t} = y_{2,t-1} + \varepsilon_{2t} \\ y_{3t} = y_{3,t-1} + \varepsilon_{3t} \end{cases} \tag{2.7}$$

where $y_{1t} = a_{2t}y_{2t} + a_{3t}y_{3t} + u_t$ is the cointegrating regression along with a_{2t} , a_{3t} known. We usually describe the cointegration system as the way like (2.7) when for each time series, there is

$$\begin{cases} y_{1t} = y_{1,t-1} + \varepsilon_{1t} \\ y_{2t} = y_{2,t-1} + \varepsilon_{2t} \\ y_{3t} = y_{3,t-1} + \varepsilon_{3t} \end{cases} \tag{2.8}$$

and the cointegrating regression is $y_{1t} = a_{2t}y_{2t} + a_{3t}y_{3t} + u_t$. Furthermore, it is supposed that

$$\varepsilon_t = \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{pmatrix} \sim nid \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ & \sigma_2^2 & \sigma_{23} \\ & & \sigma_3^2 \end{pmatrix} \right) \tag{2.9}$$

where σ_i^2 expresses variance of ε_{it} , σ_{ij} denotes covariance of ε_{it} and ε_{jt} , and the mean of ε_i is equal to 0.

The error term, u_t , is a crucial component in the system, because it should be stationary if the system is a cointegrated one, otherwise, it will be not stationary. Accordingly, if we do not know whether the system is cointegrated or not and we want to know, we need the model in Section 2.4 to study whether u_t is stationary or not. The approach description will be given in Section 3.

2.4 AR(1) model for residuals

In order to investigate the error term in cointegrating regression, first-order autoregression (AR(1)) model without drift for error term is applied into practice to test if the combination of all series is stationary or not. It is said that we do not have a known cointegration vector because the system will be considered as a cointegrated one if we find a nonzero known cointegration vector. Thereby, the first task now is to obtain estimated cointegration vector for getting the estimation of error terms in model (2.3), called residuals. After we provide residuals, residual-based test will be applied for testing whether there exists nonlinear cointegration in the system by using AR(1) model

$$\hat{u}_t = \rho \hat{u}_{t-1} + e_t \quad (2.10)$$

as the model for fitting \hat{u}_t . From (2.10), the property of combination which mostly interests us can be judged by the property of AR(1) model, that is, \hat{u}_t is $I(1)$ if $\rho = 1$ while $I(0)$ if $\rho < 1$.

3. RESIDUAL-BASED TEST

To know whether the system is a cointegrated one, we should test whether each of the elements of y_t is individually $I(1)$ first. It is omitted here because that can be easily achieved by using standard approaches in Hamilton (1994). Once the assumption that a unit root in each series individually is accepted, we start to construct the combination $u_t = \alpha' y_t$. Notice that if there exist nonlinear cointegration in the system, the error term u_t in cointegrating regression will be stationary. When the test under the null hypothesis of no nonlinear cointegration is needed, known cointegration vector has not been there. For nonlinear cointegration analysis, the estimated cointegration vector should be obtained at first to test whether it is reasonable for the estimated cointegration vector to be existed, on

which the residual-based test is based. Once we get the estimated cointegration vector, we can calculate the estimation of residuals, from which the residual-based test is derived. If the residuals are stationary, that is assuming the null hypothesis of no nonlinear cointegration is rejected and the regression of \hat{u}_t on \hat{u}_{t-1} will yield a coefficient that is less than 1. Otherwise, it will be a unit one. The aim of residual-based test is to test whether there exist nonlinear cointegration in the system by the residual model (2.10), which is equivalent to testing if residuals is $I(0)$. Since we set up the null hypothesis against alternative hypothesis:

$$\begin{aligned} H_0 : \rho &= 1 \\ H_1 : \rho &< 1 \end{aligned} \quad (3.1)$$

which means

$$\begin{aligned} H_0 : \hat{u}_t &\sim I(1) \\ H_1 : \hat{u}_t &\sim I(0) \end{aligned} \quad (3.2)$$

according to model (2.10).

To construct a statistic for hypothesis test in residual-based test, one question is noticed in our case. The cointegration vector is unknown. If the theoretical model of the system dynamics does suggest a particular value for the cointegration vector, standard unit root tests such as Phillips's Z_ρ test could be considered. However, if the theoretical model of the system dynamics does not suggest a particular value for the cointegration vector, standard unit root tests should be considered on the estimated residuals. Although these test statistics are constructed in the same way as the one when they are applied to the residuals testing from a known cointegration vector, the critical values that are used to interpret the test statistics are different from the tables given by related literatures. For this kind of situation, we derive our own table of critical values of four model styles that are requested in our application case in following Section 5.

From (2.10), the residual \hat{u}_t is regressed on its own lagged value \hat{u}_{t-1} without a constant term. It is easy to yield the *OLS* estimate of coefficient ρ that is a function of \hat{u}_t

$$\hat{\rho}_T = \frac{\sum_{t=2}^T \hat{u}_{t-1} \hat{u}_t}{\sum_{t=2}^T \hat{u}_{t-1}^2} . \quad (3.3)$$

According to the alternative hypothesis that only need to consider left-hand area which is set by the meaning of testing, the residual-based test

will reject the null hypothesis, there is no nonlinear cointegration, if the value of statistics we calculate by data is less than the critical value in Table 1.

4. ESTIMATION

What we talk about mostly in residual-based test is the estimation of residuals from the estimated cointegration vector in (2.3) whose coefficients should be estimated by nonlinear estimate approach. To estimate the parameters in cointegrating regression (2.3), conditional least squares approach can be applied into practice. The problem is to minimize

$$\sum_{t=1}^T u_t^2 = \sum_{t=1}^T (y_{1t} - a_{1t}y_{1t} - a_{2t}y_{2t} - \dots - a_{nt}y_{nt})^2. \quad (4.1)$$

The theory under this kind of nonlinear least square is based on maximum likelihood estimation (MLE), while the practical approach, based on Gauss-Newton algorithm is always used in many software of statistical computing.

The joint estimation of parameters in each a_{it} is difficult because a nonlinear optimization may not converge at all. However, it does not matter for calculating the statistics and testing because it seems the value of Z_ρ statistic cannot be obviously impacted by unstable estimation. Teräsvirta (1994) suggests that it may be workable to rescale the parameters and first standardize y_{it} by dividing them by $\hat{\sigma}(y_i)$ respectively. In common sense, there are some useful experience, such as keeping some of parameters fixed and using a grid of values to estimate, but a good joint estimation depends on data and situation in practice.

Another problem of estimation for nonlinear model is also not easy to be solved, that is the initial value of parameters for Gauss-Newton algorithm. It has greatly impact on the estimation and the number of iterations, for this kind of reasons, it is important to find out an appropriate initial value for estimation. Unfortunately, there is not fixed methods for solving the problem, sometimes only depending on own experience or personal sense.

5. SIMULATION

Simulation is based on the theory of Monte Carlo method, which depends on repeated computation and pseudorandom numbers produced by com-

Table 1: Critical Values for the Phillips Z_ρ Statistic When Cointegration Vector is Unknown.

n-1	Sample size (T)	Probability that Z_ρ is less than entry						
		0.010	0.025	0.050	0.075	0.100	0.125	0.150
Model (6.1)								
2	100	-45.53	-39.13	-33.90	-30.91	-28.39	-26.73	-25.22
2	200	-44.77	-38.18	-32.48	-29.53	-27.40	-25.64	-24.18
2	500	-39.37	-33.52	-28.85	-26.30	-24.33	-22.65	-21.38
Model (6.2)								
2	100	-45.78	-39.09	-33.76	-30.43	-28.09	-26.18	-24.73
2	200	-43.18	-37.08	-32.08	-29.13	-27.16	-25.37	-23.96
2	500	-37.91	-32.63	-27.82	-25.22	-23.24	-21.72	-20.36
Model (6.3)								
2	100	-44.97	-38.32	-33.20	-30.36	-28.25	-26.80	-25.24
2	200	-43.88	-38.03	-32.78	-29.75	-27.45	-25.66	-24.19
2	500	-39.42	-33.74	-29.03	-26.14	-24.10	-22.53	-21.27
Model (6.4)								
2	100	-28.16	-24.37	-21.30	-19.77	-18.48	-17.25	-16.47
2	200	-30.97	-27.21	-23.80	-21.70	-20.17	-18.88	-17.88
2	500	-31.55	-26.86	-23.65	-21.53	-19.94	-18.70	-17.54

The probability shown at the head of the column is the area in the left-hand tail.

puters in our case¹. However, simulation results should service for application data. The simulation result of critical values for testing with different sample sizes is presented in Table 1. There are three series to be considered just as (2.7) in our PPP data, accordingly, the number of regressors, $(n - 1)$, will be only two. The cointegrating regression contained in our simulation will range on four different transition variables in four models (6.1), (6.2), (6.3) and (6.4). Small sample size is not included into Table 1 because it is not leading to a good estimation that the amount of parameters is too large for small sample. The probabilities that statistics is less than entry in Table 1 are all no more than 0.500, while the reason that we just need left-hand hypothesis was referred before.

To get the table, we first produce vector time series of $I(1)$ without drift, then estimate the parameters in the cointegrating regression such as (6.1), (6.2), (6.3) and (6.4). Sequentially the estimation of residuals in cointegrating regression is obtained for residual-based test by model (2.10). By

¹All statistical computing are in R version 2.6.0.

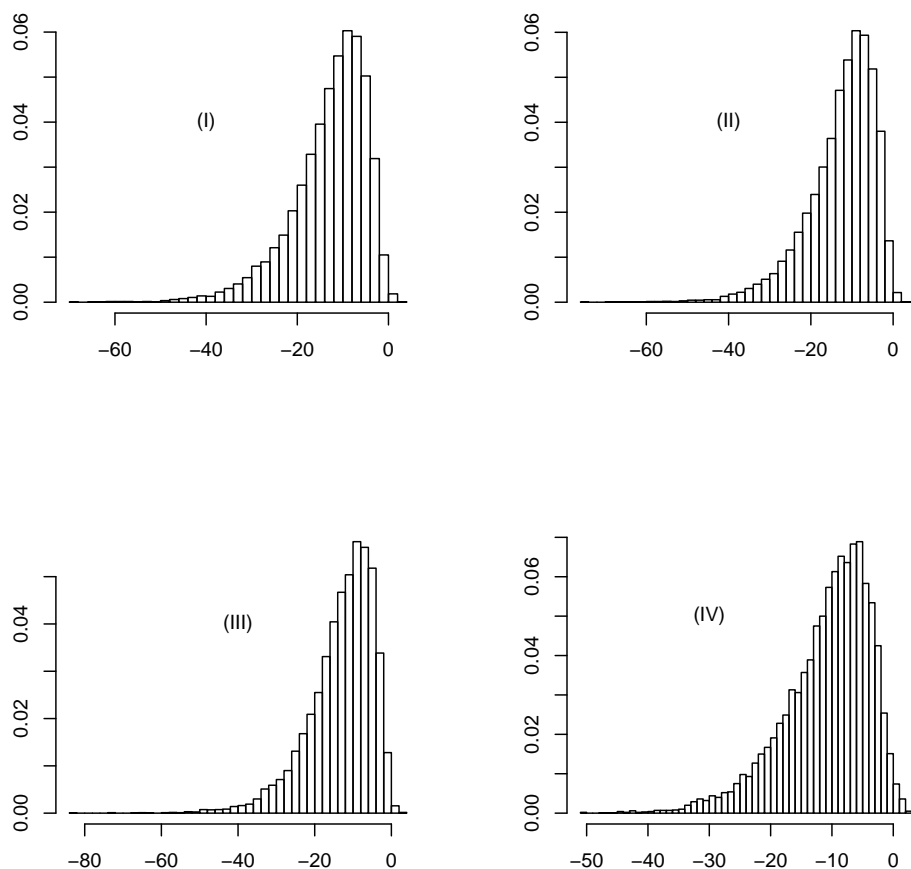


Figure 2: Asymptotic distributions of Phillips Z_ρ statistic under model (6.1), (6.2), (6.3) and (6.4) respectively.

this way, we can calculate one value of test statistic in the procedure for each model. In our simulation, we repeat the process $T = 10000$ times under different sample size to calculate the quantile values in common use in Table 1 and obtain the asymptotic distribution of the statistic with sample size being 500 in Figure 2.

In order to ensure the iteration procedure efficient when the estimation of parameters is calculated by computer, the initial value of slope parameter γ_i should be set up to be a very large one such as 1000, because if there are so many times of invalid iteration during the whole process of simulation, the critical value in simulation results will be not useful for testing opinion.

6. APPLICATION

6.1 Data

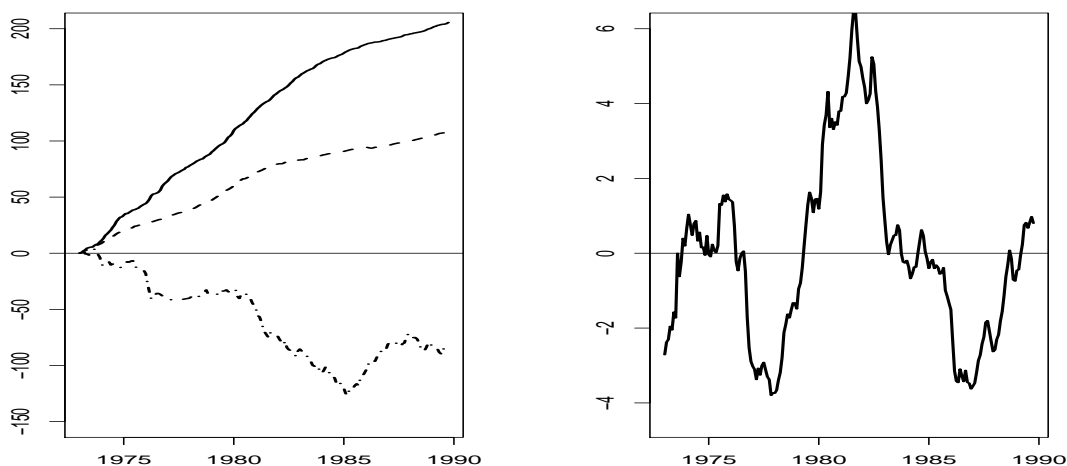


Figure 3: Normalized PPPs data for nonlinear cointegration analysis stands left and the residuals of linear cointegration in Hamilton (1994) is on right-hand side.

The Purchasing Power Parities (PPPs) data, referred in Hamilton (1994) in Figure 3 which there seems not exist linear cointegration in the system. We are eager to know whether there exist nonlinear cointegration even though there is no linear cointegration. That is the issue we are caring about.

Figure 3 plots normalized monthly PPPs data from 1973:1 to 1989:10 of the price level of United States (2) and Italy (1) along with dollar-lira exchange rate (3). Natural logs of the raw data were taken and multiplied by 100 to normalize the initial value 1973:1 of each series to be zero so that the graph is easy to read. When the simulation is launched, we usually set up the initial value of each series equal to zero. If the series are nonlinear cointegration process, the new series produced from the combination of those series will be stationary. Standing at this point, we have chances for advanced analysis. Certainly, advanced analysis is provided by the assumption that each series has been believed to be $I(1)$ individually. Actually, this part has been studied in Hamilton (1994). Here, what we are interested is whether there exists nonlinear cointegration between

these three time series based on the assumption of $I(1)$. Hence, the following work will mainly concentrate on cointegrating regression selected for residual-based test.

6.2 Model selection

According to the sections above, we first estimate the parameters in cointegrating regression model to obtain the value of residuals, \hat{u}_t , in figures. The aim for us is to test whether the residuals are stationary or not based on our data. Calculating the value of Phillips statistic, which is a function of $\hat{\rho}$, compared with critical values in Table 1, we can get the result if it is reasonable to reject the null hypothesis which means if we can believe there is nonlinear cointegration in our system.

In practice, model selection depends on the real meaning of variables, according to the aim and the main idea of cointegrating regression model of this essay, there are four models getting ready for being selected:

$$y_{1t} = a_{2t}y_{2t} + a_{3t}y_{3t} + u_t, \quad (6.1)$$

$$y_{1t} = \alpha + a_{2t}y_{2t} + a_{3t}y_{3t} + u_t, \quad (6.2)$$

$$y_{1t} = a_{2t}y_{2t} + a_{3t}y_{3t} + \beta_{21}\Delta y_{2,t-1} + \dots + \beta_{2p}\Delta y_{2,t-p} \\ + \beta_{31}\Delta y_{3,t-1} + \dots + \beta_{3p}\Delta y_{3,t-p} + u_t, \quad (6.3)$$

$$y_{1t} = \alpha + a_{2t}y_{2t} + \beta y_{3t} + u_t \quad (6.4)$$

including four different transition variables in Section 2.2 considered for each of them.

The first selection is based on the residual figures. From the residual figures of each model for primary selecting, we choose the interested models whose residual figure leads that the residual is stationary. If the residual from one model shows obvious time trend, we will skip that model directly, since we design more than 16 models to be included for finding a model that fits our data much better.

By this way, four of them are picked up presented in Table 2 according to Figure 4, given estimated models for cointegrating regression (6.5), (6.6), (6.7) and (6.8) at the end of this section. From the left-hand drawing in Figure 3, we can discern that there is obvious difference between these two figures of residuals.

The next step requires us to use statistic to give test results theoretically. For those four models (6.5), (6.6), (6.7) and (6.8), the estimators of parameter ρ are 0.85, 0.83, 0.83, 0.89 and the values of statistic Z_ρ are -32.7,

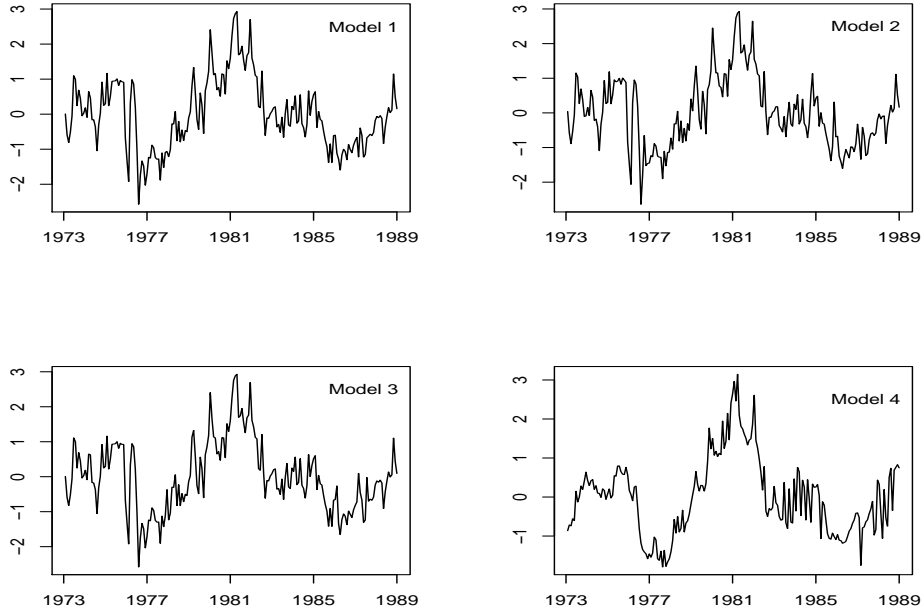


Figure 4: Residuals by Model (6.5), (6.6), (6.7) and (6.8).

Table 2: First Model Selection.

Model	Transition variable
$y_{1t} = a_{2t}y_{2t} + a_{3t}y_{3t}$ $+ \beta_{21}\Delta y_{2,t-1} + \dots + \beta_{2p}\Delta y_{2,t-p}$ $+ \beta_{31}\Delta y_{3,t-1} + \dots + \beta_{3p}\Delta y_{3,t-p}$ $+ u_t$	$(\Delta y_{i,t-1} - c_1)(\Delta y_{j,t-1} - c_2), i \neq j$ $(\Delta y_{i,t-1} - c_i)$ $(\Delta y_{i,t-1} - c_1)(\Delta y_{i,t-1} - c_2)$
$y_{1t} = \alpha + a_{2t}y_{2t} + \beta y_{3t} + u_t$	$(\Delta y_{i,t-1} - c_1)(\Delta y_{j,t-1} - c_2), i \neq j$

-34.6, -33.9,-22.0, respectively. Compared with critical values in Table 1, first three models can reject the null hypothesis of no nonlinear cointegration under the 7.5% critical and the 5% critical value, which means it is reasonable to believe there is nonlinear cointegration in the system. Obviously, (6.6) and (6.7) are much better from the results in Table 3. Although model (6.8) explains the relationship of nonlinear cointegration in our system by a much simpler way, it is not significant. For the aim of testing nonlinear cointegration, we consider that there is nonlinear cointegration in our system.

Table 3: Testing results.

Estimated model	$\hat{\rho}$	Z_ρ	p - value	α	
				0.075	0.050
(6.5)	0.85	-32.7	0.042	-29.53 *	-32.48 *
(6.6)	0.83	-34.6	0.022	-29.13 *	-32.08 *
(6.7)	0.83	-33.9	0.025	-29.75 *	-32.78 *
(6.8)	0.89	-22.0	0.075	-21.70	-23.08

"*" denotes null hypothesis can be rejected.

$$\begin{aligned}
 y_{1t} = & \left(0.55 + 0.002 (1 + \exp(-50 (\Delta y_{2,t-1} - 0.48) (\Delta y_{3,t-1} - 0.98)))^{-1}\right) y_{2t} \\
 & + \left(-0.16 + 0.09 (1 + \exp(-100 (\Delta y_{2,t-1} - 0.62) (\Delta y_{3,t-1} - 0.62)))^{-1}\right) y_{3t} \\
 & + 1.45 \Delta y_{2,t-1} + 0.24 \Delta y_{3,t-1} \\
 & + \hat{u}_t
 \end{aligned} \tag{6.5}$$

$$\begin{aligned}
 y_{1t} = & \left(-0.48 + 1.02 (1 + \exp(-50 (\Delta y_{2,t-1} - 0.50)))^{-1}\right) y_{2t} \\
 & + \left(-0.07 + 0.004 (1 + \exp(-100 (\Delta y_{3,t-1} - 0.52)))^{-1}\right) y_{3t} \\
 & + 1.46 \Delta y_{2,t-1} + 0.24 \Delta y_{3,t-1} \\
 & + \hat{u}_t
 \end{aligned} \tag{6.6}$$

$$\begin{aligned}
 y_{1t} = & \left(-0.43 + 0.97 (1 + \exp(-50 (\Delta y_{2,t-1} - 0.65) (\Delta y_{2,t-1} - 1.93)))^{-1}\right) y_{2t} \\
 & + \left(-0.14 + 0.08 (1 + \exp(-100 (\Delta y_{3,t-1} + 0.33) (\Delta y_{3,t-1} + 0.33)))^{-1}\right) y_{3t} \\
 & + 1.47 \Delta y_{2,t-1} + 0.28 \Delta y_{3,t-1} \\
 & + \hat{u}_t
 \end{aligned} \tag{6.7}$$

$$\begin{aligned}
 y_{1t} = & 2.36 + \left(0.54 - 0.01 (1 + \exp(-50 (\Delta y_{2,t-1} + 0.77) (\Delta y_{3,t-1} + 2.95)))^{-1}\right) y_{2t} \\
 & - 0.06 y_{3t} \\
 & + \hat{u}_t
 \end{aligned} \tag{6.8}$$

7. DISCUSSION

From the example analyzing the same economic data with Hamilton (1994) in this paper, we find that as the society all over the world becoming more

and more intricate, the relationship of all kinds of economic phenomenon is to be understood harder and harder. Just as our example, we could not find out cointegration in the system when the analysis focused only on linear relation in 1994, while with the developing of time series econometrics, we find out nonlinear cointegration exists in the system.

But the estimation for nonlinear cointegrating regression becomes a problem. How can we test the estimator of parameters whether it is significant or not? Although estimation may fluctuate dramatically, it has slight influence on the statistic, however, can we use the estimator for residuals if it is not significant? Another thing is that estimate of parameter depends on the given initial value of parameter during the procedure of estimating in which whatever method has been used for estimate and it is strongly dependent. Some values can give good estimation while some cannot.

When we do our simulation, the initial values of parameters are set up to be fixed during the repeating procedure. There appears a question for estimation that different simulation data need different initial values for good estimation, however, we cannot carry out like this in practice since simulation means so many times. Actually, one thing that we can improve is that fix the initial value of critical parameter, if has, to be in the suitable range, for example, set γ equal to 1000 to make things better.

References

- [1] Engle R.F. & Granger C.W.J., 1987. Co-Integration and Error Correction: Representation, Estimation, and Testing. *Econometrica*, 55(2), p.251-276.
- [2] Engle R.F. & Kozicki S., 1993. Testing for Common Features. *Journal of Business & Economic Statistics*, 11(4), p.369-380.
- [3] Hamilton, J.D., 1994. *Time Series Analysis*. Princeton, NJ: Princeton University Press.
- [4] Teräsvirta, T., 1994. Specification, Estimation, and Evaluation of Smooth Transition Autoregressive Models. *Journal of the American Statistical Association*, 89(425), p.208-218.