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Testing for Common Features

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Abstract

Based on Engle and Krozicki's idea, co-integration, a special case of common feature will be presented in this paper. A co-integrated economic system is a system with a common stochastic trend. The paper will describe the procedure of testing for nonstationary time series which aims at the case about unit root, linear and nonlinear co-integration. With individual series in a data set, a series of tests will be presented to find whether there is existing stochastic trend, and linear co-integration test based on spurious regression is also introduced to test the linear combination. Further, we discuss the nonlinear co-integration system with STAR model, and construct the nonlinear co-integration F test. By Monte Carlo simulation, the article simulates the asymptotic distribution of the F test. Finally, the empirical application is grounded on Purchasing Power Parity theory and the testing procedure is applied into real economic data, consumer price index and exchange rate of Sweden and the US. It is concluded that each individual series has a stochastic trend, and there does not exist a linear co-integration relation, but a nonlinear co-integration relation in the system.

Key Words: Common feature, Linear Co-integration, STAR Model, Nonlinear Co-integration, Monte Carlo Simulation, Spurious regression, Purchasing Power Parity theory.

Contents

List of Figures

1 Introduction

1.1 Background

There are various distinctive characteristics in economic time series, such as serial correlation, trends, stochastic, heteroscedasticity, skewness, kurtosis, and various other features. A variety of tests have been done to detect each of these features. In 1993, Engle and Kozicki proposed a question, whether features detected in single data series are actually shared in common. In this paper, the definition of common feature suggested by Engle and Kozicki is used. Engle and Kozicki, who gave us a thought-provoking insightful article, had discussed different situations with testing for different common feature. Due to the limitation of words and time, this paper will focus on time trend, unit root and co-integration.

This paper will present a series of classical definitions. A procedure for testing indiviual features and common features of the data set will be proposed to a set of nonstationary economics data. Engle and Krozicki (1993) gave the definition of common features as below,

Definition: A feature that is present in each of a group of series is said to be common to those series if there exists a nonzero linear combination of the series that does not have the feature.

When it comes to test for a specific common feature to a group of data, usually, the test's null hypothesis is set as the feature is common, while the alternative hypothesis is that the feature is not common. A basic condition of common feature's test is that the individual series of data should have the same feature. As a result, it should be confirmed that the specific feature is existed in each individual series of data, and then we can test whether this feature is common. My research will be based on the definition of Engle and Krozicki, which has a limitation of linearity property for testing the common feature.

When we analyze time series econometric, we find that almost all time series data cannot reach stationary in general study. It is necessary for us to analyze the properties of nonstationary data in order to study the real economic data. According to Hamilton's (1994) opinion, there are two most popular approaches to describe such trends to the nonstationary data. The first approach includes a deterministic time trend, and the other one includes a unit root process. So it will be critical to know whether nonstationarity in the data denpends on a deterministic time trend or a unit root during the research. For instance, economists concerned about whether economic recessions have permanent consequences for the future GNP, or represent temporary downturns instead. There are lots of differences between these two situations, such as forecasts, forecast errors, and dynamic multiplier. Finally, a difference between the time trend and unit root processes is the transformation that the data required to generate a stationary time series.

The transformation of deterministic time trend should remove the time trend term, and the transformation of unit root process should be differencing the data.

In this paper, the situation of unit root process is more focused, because temporary impulsion is a more familiar phenomenon in the real economics system. Especially in the background of financial crisis from 2008, the research of recession with unit root process will be further meaningful. For instance, Nelson and Plosser (1982) considered 13 macro-economics variables as unit root process in 1929's Great Depression. It would be more attractive if we continue to research the feature of unit root process in common, which means a very important case in common feature research, co-integration system. A co-integrated economic system is a system with a common stochastic trend. The definition of integration and linear co-integration was given by Engle and Granger's Nobel paper in 1987,

Definition: A series with no deterministic component which has a stationary, invertible, ARMA representation after differencing d times, is said to be integrated of order d, denoted $x_t \sim I(d)$.

Definition: The components of the vector x_t are said to be co-integrated of order d, b, denoted $x_t \sim CI(d, b)$, if,

a) all components of x_t are $I(d)$;

b) there exists a vector $\alpha \, (\neq 0)$ so that $z_t = \alpha' x_t \sim I(d - b), b > 0$. The vector α is called the co-integrating vector.

Most macro-economics varibles are considered as $I(1)$ processes, so I will lay more stress on the situation with $d = b = 1$. Similar to the common feature test, co-integration research is also based on the assumption of linearity, that we can call it linear co-integration.

However, it is difficult to satisfy the linear co-integration in the real economic environment, and a nonlinear model can fit real time series in the dynamic system. Many researches which attempted to generalize EG's cointegration into nonlinear form have been studied. As a nonlinear model, this paper will introduce one of the most popular nonlinear dynamic models, the smooth transition autoregressive (STAR) model, which nests a linear autoregressive model and contains a regime-switching structure. Recently, The well known example of this approach is to manifests itself in the concept of threshold cointegration and its smooth versions studied by Balke and Fomby (1997) and to employ a similar model to investigate the purchasing power parity by Enders and Falk (1998). Based on a nonlinear stationary system, we can generalize the definition of nonlinear co-integration.

Definition: Let $y_t = (y_{1t}, y_{2t}, ..., y_{nt})'$ be the n-dimensional random vector and for each $i = 1, 2, ...n, y_{it} \sim I(1)$. y_t is said to be nonlinear co-integration, if there exists a time-varying co-integration vector α such that $z_t = \alpha' y_t \sim I(0)$.

The co-integration vector should be under the following conditions:

(1) The first element α_{1t} is a nonzero-constant such that can be normalized as $\alpha = (1, \alpha_{2t}, ..., \alpha_{nt})'$

(2) $\alpha_{it}, i = 2, 3, ..., n$ are well defined function of a random variable S_t.

For each i, α_{it} has a logistic smooth transition: $\alpha_{it} = \alpha_i + G_{it}$, $G_{it} = (1 + \exp(-\gamma_i (S_{it} - c_i)))^{-1}$, where α_{it}, γ_i, and c_i are parameters, $\gamma_i \geqslant 0$. The transition variables S_{it} are weakly stationary or deterministic variables.

Then, α can be called a nonlinear cointegrating vector.

1.2 Structure

In my opinion, the testing procedure can be divided into three parts. Firstly, as soon as we get the economics data, we should judge whether the data is stationary. Previous study has given several methods to test if the data is stationary. For example, most of unit root tests set the alternative hypothesis as stationary. However, it seems not appropriate to use such kinds of tests in this case, because the null hypothesis of these tests is that the data is unit root process, and we cannot determine the regression model of the data in the beginning. Stationarity will be a reasonable assumption for the data, so KPSS test that the null hypothesis is stationarity is employed to test the data is stationary in the beginning. The second part is based on the result of nonstationary in the first step. After finding that the data is nonstationary, it will be more interesting to find the reason that cause nonstationarity, more accurately, we will wonder whether the data is a unit root process. Dickey-Fuller or Augmented Dickey-Fuller is used to test the individual series of the data, and determine the kind of integration process that each series belongs to. When a particular feature is found that unit root process is existed in each individual series, the co-integration test can be used to test whether the data owns a common feature of a stochastic process, which also means the data is a co-integration system. Further, if the data can not satisfy the co-integration system, we can test the null hypothesis of a linear co-integration against nonlinear co-integration. In testing part, some popular statistics are used to achieve the hypothesis, in order to get more information and a more meaningful conclusion from the data. With the real economic data of Norway and the United States, the testing procedure and PPP theory are also applied.

2 Methodology

Indeed, most economic and financial time series are better characterized by an exponential trend than a linear trend. So before researching such economic and financial data, I would like to take the natural log-transformation first.

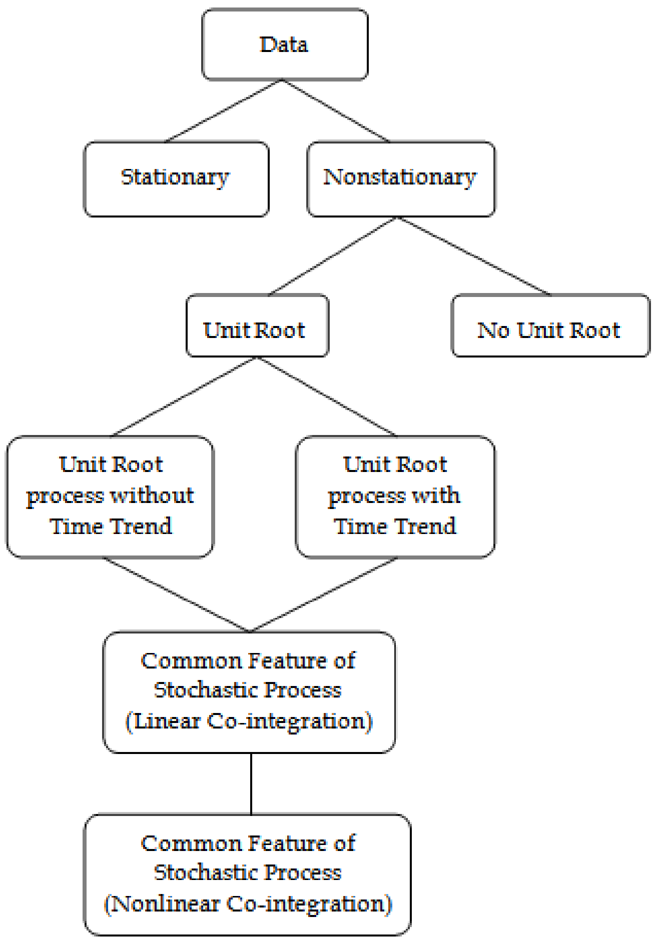

Figure 1: Stucture of testing for Co-integration system

$$\log(Y_t) = \delta t \tag{1}$$

For the nonstationary time series caused by unit root, we can do the analysis of co-integration, and find more economic information by the analysis.

In order to describe the time series tests' asymptotic distribution, Brownian Motion is introduced.

Definition: Standard Brownian motion $W(\cdot)$ is a continuous-time stochastic process, associating each date $t \in [0,1]$ with the scalar $W(t)$ such that:

a) $W(0) = 0$;

b) For any dates $0 \le t_1 \le t_2 \le ... \le t_k \le 1$, the changes $[W(t_2) - W(t_1)]$, $[W(t_3) - W(t_2)], ..., [W(t_k) - W(t_{k-1})]$ are independent multivariate Gaussian with $[W(s) - W(t)] \sim N(0, s-t)$;

c) For any given realization, $W(t)$ is continuous in t with probability 1.

2.1 Literature Review

Engle and Kozicki's (1993) article is rich with ideas, many of which call for comment. The article introduces a class of statistical tests for the hypothesis of some features. They gave a classical definition of common feature, and proposed methods to test for the common features. This paper is grouned on their research to continue my study in the field of nonstationary time series data.

Another classical article was written by Engle and Granger (1987), they discussed the linear co-integration system, and suggested one of the most popular methods, error correction model, to test the co-integration system. They pay more attention on the error correction model in order to develop estimation procedures and tests.

Johnson (1990) did an empirical application of Engle and Granger's theory to research the Purchasing Power Parity theory. The paper is a successful case of applying the co-integration system to the PPP theory with the data of Canada and the United States. He discovered that the PPP theory is maintained during both fixed and flexible exchange rate regimes.

2.2 Testing for stationarity

KPSS test is used as the first step to test the staionarity of data. In KPSS test, $\{y_t\}, t = 1, 2, ...N$ is the observed series that we will test stationarity. Assume that we can decomposes the series into the sum of a deterministic trend, a random walk, and a stationary error with the following linear regression model

$$y_t = r_t + \beta t + \varepsilon_t \tag{2}$$

where r_t is a random walk (i.e., $r_t = r_{t-1} + u_t$), and $u_t \sim i.i.d.N\left(0, \sigma_u^2\right)$; βt is a deterministic trend; ε_t is a stationary error.

To test in this model if y_t is a trend stationary process, namely, the series is stationary around a deterministic trend, the null hypothesis will be $\sigma_u^2 = 0$, which means that the intercept is a fixed constant, against the alternative of a positive σ_u^2. The KPSS test statistic is constructed as below,

$$KPSS = N^{-2} \sum_{t=1}^{N} \frac{S_t^2}{\widehat{\sigma}^2\left(p\right)} \tag{3}$$

In the formula, let the partial sum process of the ε_t as

$$S_t = \sum_{j=1}^{t} \varepsilon_j$$

The consistent estimator of $\widehat{\sigma}^2$ can be constructed from the residuals ε_t by (Newey and West, 1987),

$$\widehat{\sigma}^2\left(p\right) = \frac{1}{N} \sum_{t=1}^{N} \varepsilon_t^2 + \frac{2}{N} \sum_{j=1}^{p} W_j\left(p\right) \sum_{t=j+1}^{N} \varepsilon_t \varepsilon_{t-j} \tag{4}$$

where p is the truncation lag, $W_j\left(p\right)$ is an optional weighting function that corresponds to the choice of a spectral window. The KPSS test is an upper tailed test.

2.3 Processes with deterministic time trend

OLS estimation of the parameters of a simple time trend can be applied as,

$$y_t = \alpha + \delta t + \varepsilon_t, \varepsilon_t \sim N(0, \sigma^2) \tag{5}$$

For the OLS t test, I set the null hypothesis of the constant term $\delta = \delta_0$, and the asymptotic distribution of δ's t statistic is,

$$t_T \xrightarrow{p} \frac{T^{\frac{3}{2}}\left(\widehat{\delta}_T - \delta_0\right)}{\left[\sigma^2 \begin{pmatrix} 0 & 1 \end{pmatrix} Q^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]^{\frac{1}{2}}} = \frac{T^{\frac{3}{2}}\left(\widehat{\delta}_T - \delta_0\right)}{\sigma\sqrt{q^{22}}} \tag{6}$$

where,

$$Q \equiv \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix} \tag{7}$$

which is asymptotically distributed as N(0,1), and the value of t statistic is,

$$t_T = \frac{T^{\frac{3}{2}}\left(\widehat{\delta}_T - \delta_0\right)}{s_T\sqrt{q^{22}}} \tag{8}$$

The result of this test will provide the information to determine the case in the next test to the unit root process.

2.4 Processes with unit roots

In this part, I would like to use Dickey-Fuller and Augmented Dickey-Fuller tests as the main tests to judge whether the data is a unit root process. The Augmented Dickey-Fuller test is the extension situation that the lag is bigger than one, so the ADF's asymptotic distribution is based on the DF's. To calculate the accurate asymptotic distribution of DF and ADF, we can use the Brownian motion's properties to describe the distribution. In the discussion of DF's statistic and asymptotic distribution, I will focus on the cases of constant term without time trend and constant term with time trend.

2.4.1 Dickey-Fuller test

Dickey-Fuller test of constant term without time trend In this case, we could consider the OLS estimation of ρ based on an $AR(1)$ model,

$$y_t = \alpha + \rho y_{t-1} + u_t, u_t \sim N\left(0, \sigma^2\right) \tag{9}$$

We set the null hypothesis is $\rho = 1$ and $\alpha = 0$. To the statistics of constant term and lag term, they have different convergent rate to the asymptotic distribution. We should introduce a scaling matrix to amend the rate of convergence, when we calculate the asymptotic distribution. The scaling matrix should be,

$$\Gamma_T \equiv \begin{pmatrix} \sqrt{T} & 0 \\ 0 & T \end{pmatrix}$$

The calculation of $\rho' s$ statistic should be,

$$\rho_T = T\left(\widehat{\rho}_T - 1\right) \tag{10}$$

$$t_T = \frac{\widehat{\rho}_T - 1}{\widehat{\sigma}_{\widehat{\rho}_T}} \tag{11}$$

Dickey-Fuller test in Case 4 For Dickey-Fuller test in Case 4, the joint null hypothesis is $\rho = 1$ and $\delta = 0$, and the regression model is,

$$y_t = \alpha + \rho y_{t-1} + \delta t + u_t, u_t \sim N\left(0, \sigma^2\right) \tag{12}$$

The calculation of $\rho's$ statistic is,

$$\rho_T = T\left(\widehat{\rho}_T - 1\right) \tag{13}$$

$$t_T = \frac{\widehat{\rho}_T - 1}{\widehat{\sigma}_{\widehat{\rho}_T}} \tag{14}$$

2.4.2 Augmented Dickey-Fuller test

An alternative representation of an AR(p) process Suppose the data generated from,

$$(1 - \Phi_1 L - \Phi_2 L^2 - \ldots - \Phi_p L^p) y_t = \varepsilon_t \tag{15}$$

Define that,

$$\rho \equiv \Phi_1 + \Phi_2 + ... + \Phi_p$$

$$\varsigma_j \equiv -\left(\Phi_{j+1} + \Phi_{j+2} + ... + \Phi_{j+p}\right)$$

So,

$$\begin{aligned} & 1 - \Phi_1 L - \Phi_2 L^2 - \ldots - \Phi_p L^p \\ = \; & (1 - \rho L) - \left(\varsigma_1 L + \varsigma_2 L^2 + ... + \varsigma_{p-1} L^{p-1}\right)(1 - L) \end{aligned}$$

The equation (17) can be written equivalently as,

$$y_t = \rho y_{t-1} + \varsigma_1 \vartriangle y_{t-1} + \varsigma_2 \vartriangle y_{t-2} + ... + \varsigma_{p-1} \vartriangle y_{t-p+1} + \varepsilon_t \tag{16}$$

Augmented Dickey-Fuller test in Case 2 The regression model should be,

$$y_t = \alpha + \rho y_{t-1} + \varsigma_1 \vartriangle y_{t-1} + \varsigma_2 \vartriangle y_{t-2} + ... + \varsigma_{p-1} \vartriangle y_{t-p+1} + \varepsilon_t \tag{17}$$

The Null hypothesis is $\rho = 1$ and $\alpha = \alpha_0$, and the alternative hypothesis of ρ is $|\rho| < 1$, which means that y_t is stationary.

The statistic and asymptotic distribution to ρ are,

$$\rho_T = T\frac{\lambda}{\sigma}\left(\widehat{\rho}_T - 1\right) \xrightarrow{L} \frac{\frac{1}{2}\left\{[W(r)]^2 - 1\right\} - W(1)\int W(r)\,dr}{\int [W(r)]^2\,dr - \left[\int W(r)\,dr\right]^2} \tag{18}$$

$$t_T = \frac{\widehat{\rho}_T - 1}{\widehat{\sigma}_{\widehat{\rho}_T}} \xrightarrow{L} \frac{\frac{1}{2}\left\{[W(r)]^2 - 1\right\} - W(1)\int W(r)\,dr}{\left\{\int [W(r)]^2\,dr - \left[\int W(r)\,dr\right]^2\right\}^{\frac{1}{2}}} \tag{19}$$

which has the same asymptotic distribution with Dickey-Fuller test in case 2, and $\frac{\lambda}{\sigma} = \frac{1}{1-\varsigma_1-\varsigma_2-\ldots-\varsigma_{p-1}}$.

Augmented Dickey-Fuller test in Case 4 The regression model is,

$$y_t = \alpha + \rho y_{t-1} + \varsigma_1 \bigtriangleup y_{t-1} + \varsigma_2 \bigtriangleup y_{t-2} + \ldots + \varsigma_{p-1} \bigtriangleup y_{t-p+1} + \delta t + \varepsilon_t \tag{20}$$

The joint null hypothesis is $\rho = 1$, $\alpha = \alpha_0$ and $\delta = 0$, and the alternative hypothesis of ρ is $|\rho| < 1$.

The statistic of ρ is,

$$\rho_T = T\frac{\lambda}{\sigma}\left(\widehat{\rho}_T - 1\right) \tag{21}$$

$$t_T = \frac{\widehat{\rho}_T - 1}{\widehat{\sigma}_{\widehat{\rho}_T}} \tag{22}$$

which also has the same asymptotic distribution with Dickey-Fuller test in case 4, and $\frac{\lambda}{\sigma} = \frac{1}{1-\varsigma_1-\varsigma_2-\ldots-\varsigma_{p-1}}$

2.5 Testing for linear co-integration system

In the linear co-integration test, we can set that,

H_0: There is no linear co-integration relation in the system.

H_1: There exists linear co-integration relation in the system.

As a result, if I can prove there exists or find a linear combination that satisfies that $z_t = \alpha\prime y_t \sim I(d-b)$ is an $I(0)$ process in a critical region, $P_{H_0}(z_t\ is\ stationary) \leq 5\%$, then we will reject the null hypothesis, and accept the alternative hypothesis.

In this case, the null hypothesis of "There is no linear co-integration relation in the sys tem" is equal to that "the co-integrating vector α cannot make the $z_t = \alpha\prime y_t$ be an $I(0)$ process".

When we want to estimate the co-integrating vector, we suppose that $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)'$.

If it is known for certain that the co-integrating vector has a nonzero coefficient for y_{1t}, then we can normalize the parameters as $\alpha_1 = 1$, and $\alpha = (1, -\gamma_2, ..., -\gamma_n)'$.

For $z_t = \alpha\prime y_t = T^{-1}\sum_{t=1}^{T} z_t^2 = T^{-1}\sum_{t=1}^{T}(\alpha\prime y_t)^2$, if α is a co-integrating vector, then z_t should be an $I(0)$ process. As a result, the objective is to choose α as to minimize,

$$T^{-1}\sum_{t=1}^{T}(\alpha\prime y_t)^2 = T^{-1}\sum_{t=1}^{T}(y_{1t} - \gamma_2 y_{2t} - ... - \gamma_n y_{nt})^2 \tag{23}$$

And the consistent estimates of α can be obtained by OLS regression of the first variable of y_t on other variables,

$$y_{1t} = \gamma_2 y_{2t} + \gamma_3 y_{3t} + ... + \gamma_n y_{nt} + u_t \tag{24}$$

With a constant term, the regression can be also obtained by,

$$y_{1t} = \alpha + \gamma_2 y_{2t} + \gamma_3 y_{3t} + ... + \gamma_n y_{nt} + u_t \tag{25}$$

During the test for co-integration relationship, it is easy for us to get the co-integrating vector's regression of an $I(1)$ variable on several $I(1)$ variables as the regressor for which we cannot find the coefficients produce an $I(0)$ error term.

$$u_t = y_{1t} - \widehat{\alpha} - \widehat{\gamma}_2 y_{2t} - ... - \widehat{\gamma}_n y_{nt} \tag{26}$$

The regression is therefore subject to the spurious regression problem. As a result, when the co-integrating vector is unknown, but estimated, we cannot adapt the normal Dickey-Fuller test or other tests to test whether the linear combination is a unit root process. We should construct one of the standard unit root tests on the estimated residuals, such as Augmented Dickey-Fuller t test. Actually, the statistics are constructed with the same way when these tests are applied to the individual data, but the asymptotic distributions and critical values are different. The reason is that these tests will be used to the residuals from a spurious regression. We can have a wider range to select the lag in residuals, so I would like to use Augmented Dickey-Fuller t test.

To calculate the Augmented Dickey-Fuller t statistic, lagged changes in residuals could be added to the regression,

$$\widehat{u}_t = \varsigma_1 \bigtriangleup u_{t-1} + \varsigma_2 \bigtriangleup u_{t-2} + ... + \varsigma_{p-1} \bigtriangleup u_{t-p+1} + \rho\widehat{u}_{t-1} + e_t \tag{27}$$

The t statistic is using the standard OLS formula as (32).

2.6 Testing for nonlinear co-integration system

2.6.1 STAR (Smooth Transition Autoregressive) model

In this paper, I would introduce the tri-variate case, because the model will be applied to the PPP theory. The model form is as follow:

$$\begin{aligned} y_{1t} &= \alpha_{2t}y_{2t} + \alpha_{3t}y_{3t} + u_{1t} \\ y_{2t} &= y_{2,t-1} + u_{2t} \\ y_{3t} &= y_{3,t-1} + u_{3t} \end{aligned} \tag{28}$$

where,

$$u_t = \begin{pmatrix} u_{1t} \\ u_{2t} \\ u_{3t} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ & \sigma_2^2 & \sigma_{12} \\ & & \sigma_3^2 \end{pmatrix}\right) \tag{29}$$

$$\alpha_{it} = \alpha_i + G\left(\gamma_i, \triangle y_{i,t-d}, c_i\right), i = 2,3 \tag{30}$$

Here, $G\left(\gamma_i, \triangle y_{i,t-d}, c_i\right)$ is defined as the logistic smooth transition function,

$$G\left(\gamma_i, \triangle y_{i,t-d}, c_i\right) = \left(1 + \exp\left(-\gamma_i\left(\triangle y_{i,t-d} - c_i\right)\right)\right)^{-1} - \frac{1}{2}, i = 2,3 \tag{31}$$

In the model, γ_i is a parameter which can control the rate of transition from one regime to another at c_i, and the rate of transition will be higher by an increase of γ_i. The delay parameter d of transition variable, $\triangle y_{i,t-d}$, should be determined the lag d from the data. Tsay (1989) suggested that we should select the order of autoregression p first, and then determine d by varying it and choosing the value by minimizing the p value of the linearity test. This article will fix the delay parameter $d = 1$.

2.6.2 Testing for nonlinear co-integration

In the nonlinear co-integration test, we can set the null hypothesis of a linear cointegration against nonlinear cointegration, which can be expressed equivalent as below,

$$H_0 : \gamma_2 = 0, \gamma_3 = 0$$

Under the null hypothesis, the transition function $G\left(\gamma_i, \triangle y_{i,t-d}, c_i\right) = 0$, then the model can be reduced to a linear model. And we could use Taylor expansion of γ around 0 to get a auxiliary regression. Luukkonen (1988) and He & Sandberg (2005) proposed that the first-order expansion would lead a low power if the transition takes place only in the intercept. So I will use the third-order Taylor expansion to describe the asymptotic ditribution of the test,

$$G\left(\gamma_i, \triangle y_{i,t-1}, c_i\right) = \frac{\gamma_i\left(\triangle y_{i,t-1} - c_i\right)}{4} + \frac{\gamma_i^3\left(\triangle y_{i,t-1} - c_i\right)^3}{48} + r_3\left(\gamma_i\right) \quad (32)$$

Under the alterative hypothesis, the auxiliary regression is as below,

$$y_{1t} = \left(y_{2t}h_{2t}\right)' \varphi_2 + \left(y_{3t}h_{3t}\right)' \varphi_3 + u_t^* \quad (33)$$

where,

$$\begin{aligned} h_{2t} &= \left(1, \triangle y_{2,t-1}, \triangle y_{2,t-1}^2, \triangle y_{2,t-1}^3\right)' \\ h_{3t} &= \left(1, \triangle y_{3,t-1}, \triangle y_{3,t-1}^2, \triangle y_{3,t-1}^3\right)' \\ \varphi_2 &= \left(\varphi_{20}, \varphi_{21}, \varphi_{22}, \varphi_{23}\right)' \\ \varphi_3 &= \left(\varphi_{30}, \varphi_{31}, \varphi_{32}, \varphi_{33}\right)' \end{aligned}$$

In order to calculate the value of F statistic,
Restricted model:

$$p_t = \alpha_1 p_t^* + \alpha_2 s_t \quad (34)$$

Unrestricted regression:

$$\begin{aligned} p_t = \; & \varphi_{20} p_t^* + \varphi_{21} p_t^* \triangle p_{t-1}^* + \varphi_{22} p_t^* \triangle p_{t-1}^{*2} \\ & + \varphi_{23} p_t^* \triangle p_{t-1}^{*3} + \varphi_{30} s_t + \varphi_{31} s_t^* \triangle s_{t-1}^* \\ & + \varphi_{32} s_t^* \triangle s_{t-1}^{*2} + \varphi_{33} s_t^* \triangle s_{t-1}^{*3} \end{aligned} \quad (35)$$

Then, we can get,

$$F = \frac{\left(RSS_0 - RSS_1\right)/m}{RSS_1/\left(T - k\right)}$$

where RSS_0 is the summary squared residuals of restrictied model, RSS_0 is the summary squared residuals of unrestrictied model, m is the number of restricted conditions, T is the sample size, and k is the number of parameters in the unrestricted model.

As a result, the null hypothesis of auxiliary regression should be $\varphi_{ij} = 0, i = 2, 3; j > 0$.

2.6.3 Asymptotic distribution

With Monte Carlo simulation, we can get the data generating function under the null hypothesis as below,

$$\begin{aligned} y_{1t} &= \alpha_{2t} y_{2t} + \alpha_{3t} y_{3t} + u_{1t} \\ y_{2t} &= y_{2,t-1} + u_{2t} \\ y_{3t} &= y_{3,t-1} + u_{3t} \end{aligned} \tag{36}$$

where, $u_{it} \sim i.i.d.N\left(0,1\right), i = 1,2,3$

Then, with the auxiliary regression and the OLS estimation to parameters, the value of F statistic can be calculated as,

$$\begin{aligned} b_T &= \left(X'X\right)^{-1} X' y_{1t},\ X = \left(h_{2t}, h_{3t}\right) \\ s^2 &= \frac{\left(y_{1t} - X \cdot b_T\right)' \left(y_{1t} - X \cdot b_T\right)}{T-k} \\ F &= \frac{\left(R \cdot b_T - r\right)' \left(s^2 R \left(X'X\right)^{-1} R'\right)^{-1} \left(R \cdot b_T - r\right)}{m} \end{aligned}$$

where, R is a diagonal matrix and r is a vector fixed by the null hypothesis. Repeat the above process 10,000 times to simulate OLS F statistic's asymptotic distribution. The asymptotic distribution table is,

Sample Size	*Probability that F statistic is greater than entry*							
T	*0.99*	*0.975*	*0.95*	*0.90*	*0.10*	*0.05*	*0.025*	*0.01*
25	0.32	0.43	0.55	0.71	3.53	4.38	5.20	6.50
50	0.29	0.40	0.50	0.63	2.68	3.19	3.69	4.36
100	0.29	0.38	0.48	0.61	2.47	2.88	3.27	3.82
250	0.27	0.36	0.46	0.59	2.33	2.73	3.10	3.59
500	0.28	0.37	0.46	0.59	2.26	2.61	2.98	3.40

Critical Values for the Nonlinear Co-integration F statistic

3 Empirical Application to PPP theory

3.1 Data

The quarterly data from 1971:1 to 2004:4 for the consumer price indexes for the United States (p_t) and Norway (p_t^*), along with the exchange rate (s_t), where s_t is in terms of the number of US dollars needed to purchase an Norway Krone. In order to make the data comparable to the horizontal and vertical reach comparable, natural logs of the raw data were taken and multiplied by 100, and the initial value for 1971:1 was then subtracted, as in,

$$p_t = 100 \times \left[\log\left(p_t\right) - \log\left(p_{1971:1}\right)\right] \tag{37}$$

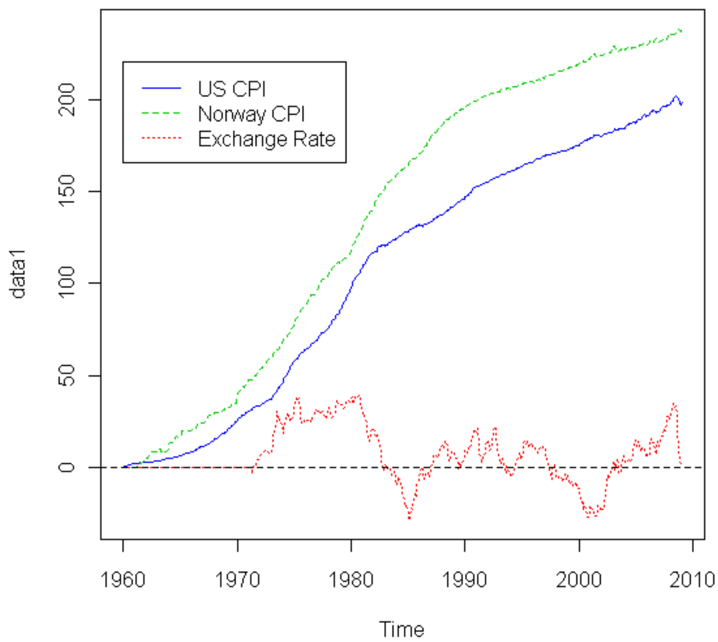

Figure 2: One hundred times the log of the price level in the US, the dollar-krona exchange rate, and the price level in Norway, monthly, 1960:01-2008:12

3.2 Testing for stationary

As the first step to test the economics time series, I use KPSS test to test whether this set of data is stationary. More accurately, the kind of stationary should be trend stationary during the test. The data is a group of quarterly data, so it will be reasonable to choose the lag as 12. The model with lag 12 has the smallest AIC value (among lag 13 and 15), so I will determine the lag is 12. With the lag is equal to 12, we can get the KPSS statistic value as below,

T=590	p_t	p_t^*	s_t	Critical value (5%)
KPSS statistic	0.64	0.73	0.23	0.146
p-value	0.01	0.01	0.01	0.05

By observing the KPSS statistics of data, it is obvious that the values of statistics are smaller than the critical value under the 5% significance. The asymptotic distribution is distributed as right-tailed, so we should reject the null hypothesis of KPSS test, that the series is trend stationary, and accept the alternative hypothesis, that the data is nonstationary.

3.3 Testing for individual series

3.3.1 Simple time trend

T=590	p_t	p_t^*	s_t	Critical value (5%)
Constant Statistic	-15.68	-0.21	14.83	± 1.96
Time Trend Statistic	767.59	643.61	-11.82	± 1.96

With the values in the table, it is obvious that we should reject the null hypothesis of $\delta = 0$. We can ensure that there exists a deterministic time trend in the individual series of data.

3.3.2 Unit root process with time trend

When we have confirmed that this set of data is belong the nonstationary data by KPSS test, it is important to find the reason or the property that cause this kind of nonstationarity further. Whether each variable of the data set is a unit root process? I will use following unit root test to answer this question. The data is composed by consumer price index and exchange rate, these time series should exhibit a deterministic time trend during the analysis of unit root, and many previous economics research papers suggest that there is a deterministic time trend in the test. Moreover, the evidence from the test of simple time trend also suggests that there should be a time trend term. Because the lag is bigger than 1, I will use the Augmented Dickey-Fuller test to test whether the data owns a unit root process.

In ADF test, the null hypothesis of $\rho = 1$. With the regression model of $y_t = \alpha + \rho y_{t-1} + \varsigma_1 \vartriangle y_{t-1} + \varsigma_2 \vartriangle y_{t-2} + ... + \varsigma_{p-1} \vartriangle y_{t-p+1} + \delta t + \varepsilon_t$, we can get the test statistics' value table as below,

T=590	p_t	p_t^*	s_t	Critical value (5%)
ADF rho statistic	-0.71	0.34	-15.28	-21.5
ADF t statistic	-0.81	0.39	-2.56	-3.42

The asymptotic distribution of ADF statistic is left-tailed distributed, so it is obvious that we cannot reject the null hypothesis of $\rho = 1$ for any of the series, which means that the data owns unit root processes. However, we still cannot get the result that the each series is an $I(1)$ process. According to the definition of integration, we can test the unit root process to the first order differencing data as a simple way. If the first order differencing data can achieve stationary, that means the original series is an $I(1)$ process. And if the first order differencing data can not achieve stationary, that means the original series is a higher integration process than $I(1)$.

T=589	p_t	p_t^*	s_t	Critical value (5%)
ADF rho statistic	-52.72	-131.60	-349.93	-20.5
ADF t statistic	-3.51	-3.81	-3.55	-3.42

All the statistic values locate in the rejection region to each test, we can reject the null hypothesis, and accept that the differencing data can achieve stationarity. Each original series of the data is actually an $I(1)$ process. Three individual series are $I(1)$ processes, and they all own the stochastic process.

3.4 Testing for linear co-integration

By the test analysis above, it is obvious that the individual series of data own the property of unit root process. Now, we are more interested in knowing whether these individual data own the common feature of these two properties. Actually, whether the data is the co-integration system is more meaningful in economics. We can use the Purchasing Power Parity theory as the economics theory to support the next test.

The PPP relationship is,

$$y_t = (p_t, s_t, p_t^*)' \tag{38}$$

$$\alpha = (\alpha_1, -\alpha_2, -\alpha_3)' \tag{39}$$

Where the variables are,

p_t = foreign currency (American dollar) price index ;
p_t^* = domestic currency (Sweden krone) price index;
s_t = the exchange rate, the domestic currency (Swedish krone) price of a unit of foreign currency (American dollar).

By normalizing the first element of α, the normalized vector α is defined as,

$$\alpha = \begin{pmatrix} \alpha_1/\alpha_1 \\ -\alpha_2/\alpha_1 \\ -\alpha_3/\alpha_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -\gamma_2 \\ -\gamma_3 \end{pmatrix} \tag{40}$$

And in PPP theory, with $\alpha = (1, -\gamma_2, -\gamma_3)'$, $u_t = p_t - c - \gamma_2 s_t - \gamma_3 p_t^*$ is defined as the real exchange rate.

In order to estimate the co-integrating vector, we can regress $p_t = c + \gamma_2 s_t + \gamma_3 p_t^* + u_t$,

$$p_t = -7.0968 + 0.8294 s_t + 0.0261 p_t^*$$

Then we can generate the real exchange rate series with the definition of $u_t = p_t - c - \gamma_2 s_t - \gamma_3 p_t^*$.

Next step is to test whether u_t is stationary. We should determine the lag in the regression models. Because the new series is generated from the data of monthly data, so it is also reasonable to suspect that the lag is equal to 12. And I will compare the values of AIC to the models with different lags around 12 (such as 11, 12, 13), and determine the regression model with the information from AIC. Finally, I found that AIC value of the model with lag 12 was not the smallest, but the difference with other models is less than 1%. So the lag is still equal to 12.

The co-integrating vector is unknown and estimated, and the regression is a spurious regression problem, so we cannot use the Dickey-Fuller test directly. Augmented Dickey-Fuller t test is a standard unit root tests on the estimated residuals. We can get the testing value as below,

T=590	u_t	u_{t1} (Differencing)	Critical value (5%)
ADF t statistic	-2.55	-4.21	-3.77

In the table, u_{t1} means the differencing series of u_t. It obviously that the linear combination u_t is not stationary, not an $I(0)$ process, but an $I(1)$ process, which means that the linear combination has the same property of individual series. The stochastic process is not a common feature in this group of data, because the effect of each individual series' feature can not be removed by the linear combination. We should reject the null hypothesis that the system has a co-integration relation, and accept the alternative hypothesis, there is no co-integration relation in this system. This also shows an opposite evidence to prove the Purchasing Power Parity theory.

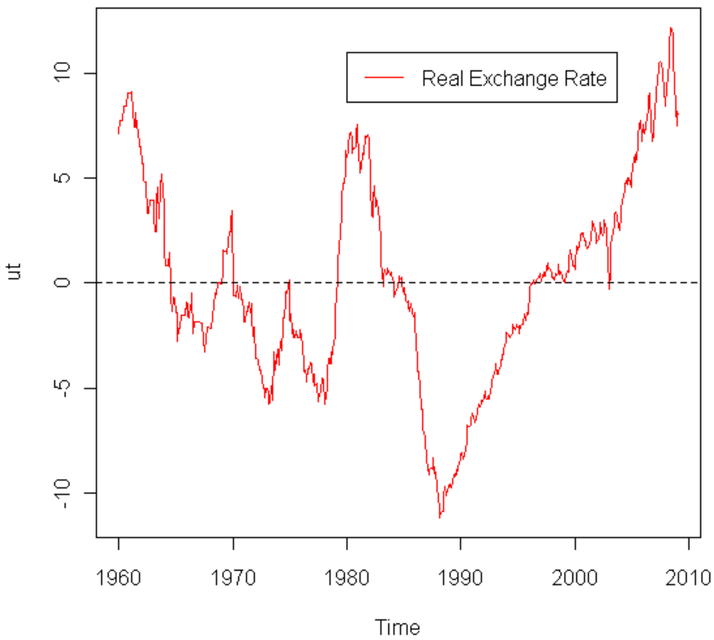

Figure 3: The real dollar-krona exchange rate, monthly, 1960:01-2008:12

3.5 Testing for nonlinear co-integration

In this test, we can set the null hypothesis as, H_0: $\gamma_1 = \gamma_2 = 0$. Under the auxiliary regression of third order Taylor expansion, we apply OLS to fit restricted regression and non-restricted regression.

The value of F statistic is $F = \frac{(RSS_0 - RSS_1)/m}{RSS_1/(T-k)} = 3.09 > 2.61$, which is the critical value of F in 5% significance. So we can reject the null hypothesis of nonlinear co-integration F test, which means that there is a nonlinear co-integration relation in the system.

4 Conclusion and Discussion

In this paper, based on the discussion of common feature in time series, we propose a procedure of how to test the properties of nonstationary time series data including the properties of individual series, the linear and nonlinear combination. As a special case of common feature, linear and nonlinear co-integration are the main points that I have discussed in the article. Co-integration system has a feature of common stochastic process, and unit root process is not only a part of co-integration, but also a very significant feature in the real economics area. Once the data can be satisfied by the test for stationarity, unit root process, and linear or nonlinear co-integration, a stochastic process can be considered as a common feature of a data group. About the co-integration system, the purpose is to find a linear or nonlinear combination that can reduce the impact of stochastic process. The article discuss the tests for common features by the case of co-integration, and we generalize the definition and test of linear co-integration to nonlinear co-integration system. Similarly, we can also generalize the common feature under the linear assumption to a nonlinear assuption.

The testing procedure is applied to PPP theory and the real time series data, Norway CPI, US CPI, as well as the exchange rate between Norway and the US. I get the result that each series of the data, Norway CPI, US CPI, and the exchange rate, is a nonstationary and $I(1)$ process. However, when I test the common feature of stochastic process to the combination of these series, I find that there is no linear co-integration relation, but a nonlinear co-integration relation in the system.

When we find the asymptotic distributions for the nonlinear co-integration F test, we use the Monte Carlo simulation to get the asymptotic distribution. However, the simulation is not the accurate asymptotic ditribution of test. If we can get the theoretical distribution of nonlinear co-integration F test, the result of test will be more reliable. In this paper, we only try to use the nonlinear model, STAR model, to fit the data. Although the data can be fitted well, it cannot show that nonlinear co-integration test is more efficient than the linear one. We consider co-integration as a very important and special case in common features, and we introduce the procedure of testing for co-integration in detail. However, there are lots of other common features we can research, such as skewness, kurtosis, and heteroskedasticity, and if we can study more case of common features in detail, we can get more information to explain the economic phenomenon.

Part I
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Part II
Appendices

The aysmptotic distribution of process with deterministic time trend

In order to find the asymptotic distributions for regressions, we can rewrite the above model to the standard regression model,

$$y_t = x_t' + \varepsilon_t$$

where

$$x_t' \equiv \begin{pmatrix} 1 & t \end{pmatrix}$$

$$\beta \equiv \begin{pmatrix} \alpha \\ \delta \end{pmatrix}$$

Let b_t denote the OLS estimate of β based on a sample of size T:

$$b_t \equiv \begin{pmatrix} \widehat{\alpha}_T \\ \widehat{\delta}_T \end{pmatrix} = \left(\sum_{t=1}^{T} x_t x_t' \right)^{-1} \left(\sum_{t=1}^{T} x_t y_t \right)$$

So the deviation of the OLS estimate from the true value can be expressed as,

$$(b_t - \beta) = \left(\sum_{t=1}^{T} x_t x_t' \right)^{-1} \left(\sum_{t=1}^{T} x_t \varepsilon_t \right)$$

Because the OLS estimates $\widehat{\alpha}_T$ and $\widehat{\delta}_T$ have different asymptotic rates of convergence, $\widehat{\alpha}_T$ should be multiplied by $\sqrt{T}$, whereas $\widehat{\delta}_T$ should be multiplied by $T^{\frac{3}{2}}$, in order to arrive at limiting distributions. We can adjust by the matrix,

$$\Gamma_T \equiv \begin{pmatrix} \sqrt{T} & 0 \\ 0 & T^{\frac{3}{2}} \end{pmatrix}$$

resulting in,

$$\begin{pmatrix} \sqrt{T} \left(\widehat{\alpha}_T - \alpha \right) \\ T^{\frac{3}{2}} \left(\widehat{\delta}_T - \delta \right) \end{pmatrix} = \left[\Gamma_T^{-1} \left(\sum_{t=1}^{T} x_t x_t' \right) \Gamma_T^{-1} \right]^{-1} \left[\Gamma_T^{-1} \left(\sum_{t=1}^{T} x_t \varepsilon_t \right) \right]$$

Substituting the first term of (7),

$$\left[\Gamma_T^{-1}\left(\sum_{t=1}^{T} x_t x_t'\right)\Gamma_T\right] = \begin{pmatrix} T^{-1}\sum 1 & T^{-2}\sum t \\ T^{-2}\sum t & T^{-3}\sum t^2 \end{pmatrix} \to Q$$

and,

$$Q \equiv \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}$$

The second term is,

$$\Gamma_T^{-1}\left(\sum_{t=1}^{T} x_t \varepsilon_t\right) = \begin{pmatrix} \frac{1}{\sqrt{T}}\sum \varepsilon_t \\ \frac{1}{\sqrt{T}}\sum\left(\frac{t}{T}\right)\varepsilon_t \end{pmatrix}$$

ε_t is i.i.d. with mean zero, variance σ^2, and finite fourth moment. Then the first element of the vector in equation (10) satisfies,

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_t \xrightarrow{L} N\left(0, \sigma^2\right)$$

By the property of martingale difference sequence, we can get that any linear combination of the two elements in the vector in (10) is asymptotically Gaussian,

$$\begin{pmatrix} \frac{1}{\sqrt{T}}\sum \varepsilon_t \\ \frac{1}{\sqrt{T}}\sum\left(\frac{t}{T}\right)\varepsilon_t \end{pmatrix} \xrightarrow{L} N\left(0, \sigma^2 Q\right)$$

Then, the asymptotic distribution of (7) can be calculated,

$$\begin{pmatrix} \sqrt{T}\left(\widehat{\alpha}_T - \alpha\right) \\ T^{\frac{3}{2}}\left(\widehat{\delta}_T - \delta\right) \end{pmatrix} \xrightarrow{L} N\left(0, Q^{-1}\sigma^2 Q Q^{-1}\right) = N\left(0, \sigma^2 Q^{-1}\right)$$

In the process of hypothesis test for the simple time trend model, testing model for the simple time trend is,

$$y_t = \alpha + \delta t + \varepsilon_t, \varepsilon_t \sim N\left(0, \sigma^2\right)$$

For the OLS t test, I set the null hypothesis of the constant term $\alpha = \alpha_0$, and the t statistic of α is,

$$t_T = \frac{\widehat{\alpha}_T - \alpha_0}{\left[s_T^2 \begin{pmatrix} 1 & 0 \end{pmatrix} \left(X_T' X_T\right)^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]^{\frac{1}{2}}}$$

where s_T^2 denotes the OLS estimate of σ^2,

$$s_T^2 = \frac{1}{T-2}\sum_{t=1}^{T}\left(y_t - \widehat{\alpha}_T - \widehat{\delta}_T t\right)^2 \xrightarrow{p} \sigma^2$$

$$X_T' X_T = \sum_{t=1}^{T} x_t x_t'$$

For

$$\Gamma_T \equiv \begin{pmatrix} \sqrt{T} & 0 \\ 0 & T^{\frac{3}{2}} \end{pmatrix}$$

$$t_T = \frac{\widehat{\alpha}_T - \alpha_0}{\left[s_T^2 \begin{pmatrix} 1 & 0 \end{pmatrix} \left(X_T' X_T\right)^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]^{\frac{1}{2}}}$$

$$\Gamma_T^{-1}\left(X_T' X_T\right)\Gamma_T^{-1} = \left[\Gamma_T \left(X_T' X_T\right)^{-1} \Gamma_T\right]^{-1} \to Q^{-1}$$

So

$$t_T \xrightarrow{p} \frac{\sqrt{T}\left(\widehat{\alpha}_T - \alpha_0\right)}{\left[\sigma^2 \begin{pmatrix} 1 & 0 \end{pmatrix} Q^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]^{\frac{1}{2}}} = \frac{\sqrt{T}\left(\widehat{\alpha}_T - \alpha_0\right)}{\sigma\sqrt{q^{11}}}$$

q^{11} represents the (1,1) element of Q^{-1}, and t_T is an asymptotically Gaussian variable on its square root of variance, so it has a $N(0,1)$ distribution. And the value of t statistic should be,

$$t_T = \frac{\sqrt{T}\left(\widehat{\alpha}_T - \alpha_0\right)}{s_T\sqrt{q^{11}}}$$

Asymptotic distribution of Dickey-Fuller test statistic (unit root process without time trend)

The asymptotic distribution of statistic is,

$$\Gamma_T\left(b_t - \beta\right) \xrightarrow{L} \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix} \begin{bmatrix} 1 & \int W(r)\,dr \\ \int W(r)\,dr & \int [W(r)]^2\,dr \end{bmatrix}^{-1} \begin{pmatrix} W(1) \\ \frac{1}{2}\left\{[W(r)]^2 - 1\right\} \end{pmatrix}$$

and,

$$\rho_T = T\left(\widehat{\rho}_T - 1\right) \xrightarrow{L} \frac{\frac{1}{2}\left\{[W(r)]^2 - 1\right\} - W(1)\int W(r)\,dr}{\int [W(r)]^2\,dr - \left[\int W(r)\,dr\right]^2}$$

$$t_T = \frac{\widehat{\rho}_T - 1}{\widehat{\sigma}_{\widehat{\rho}_T}} \xrightarrow{L} \frac{\frac{1}{2}\left\{[W(r)]^2 - 1\right\} - W(1)\int W(r)\,dr}{\left\{\int [W(r)]^2\,dr - \left[\int W(r)\,dr\right]^2\right\}^{\frac{1}{2}}}$$

A Transformation of regression model in Case 4

In order to calculate the asymptotic distribution of DF test in case 4, we need a transformation to the regression model. So the regression model can equivalently be written as,

$$\begin{aligned} y_t &= \alpha^* + \rho^* y_{t-1} + \delta^* t + u_t \\ &= (1-\rho)\,\alpha + \rho\,[y_{t-1} - \alpha(t-1)] + (\delta + \rho\alpha)\,t + u_t \end{aligned}$$

where,

$$\alpha^* = (1-\rho)\,\alpha$$

$$\rho^* = \rho$$

$$\delta^* = \delta + \rho\alpha$$

The null hypothesis is $\rho = 1$, $\alpha = \alpha_0$ and $\delta = 0$. And in the transformed system, the null hypothesis is $\alpha^* = 0$, $\rho^* = 1$ and $\delta^* = \alpha_0$. In this case, the scaling matrix is,

$$\Gamma_T \equiv \begin{pmatrix} T^{\frac{1}{2}} & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & T^{\frac{3}{2}} \end{pmatrix}$$

Asymptotic distribution of Dickey-Fuller test statistic (unit root process with time trend)

The asymptotic distributions of statistic are,

$$\Gamma_T (b_t - \beta) = \begin{pmatrix} T^{\frac{1}{2}} \alpha^* \\ T(\rho^* - 1) \\ T^{\frac{3}{2}} (\delta^* - \alpha_0) \end{pmatrix}$$

$$\xrightarrow{L} \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sigma \end{pmatrix} \begin{bmatrix} 1 & \int W(r)\,dr & \frac{1}{2} \\ \int W(r)\,dr & \int [W(r)]^2\,dr & \int W(r)\,dr \\ \frac{1}{2} & \int W(r)\,dr & \frac{1}{3} \end{bmatrix}^{-1} \begin{bmatrix} W(1) \\ \frac{1}{2}\left\{[W(r)]^2 - 1\right\} \\ W(1) - \int W(r)\,dr \end{bmatrix}$$

$$T^2\widehat{\sigma}^2_{\widehat{\rho}_T} \xrightarrow{L} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{bmatrix} 1 & \int W(r)\,dr & \frac{1}{2} \\ \int W(r)\,dr & \int [W(r)]^2\,dr & \int W(r)\,dr \\ \frac{1}{2} & \int W(r)\,dr & \frac{1}{3} \end{bmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \equiv Q$$

$$t_T = \frac{\widehat{\rho}_T - 1}{\widehat{\sigma}_{\widehat{\rho}_T}} \xrightarrow{L} \frac{T(\widehat{\rho}_T - 1)}{\sqrt{Q}}$$