A Copula Based GARCH Dependence Model of Shanghai and Shenzhen Stock Markets

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Abstract

Copula is a function which can link two or more marginal distributions together to form a joint distribution. This paper aims to analyze the dependence between Shanghai and Shenzhen stock markets using copula theory based on GARCH. We use the synchronous 100 times daily returns data and copula based GARCH to model the joint distribution of stock index returns because copula based GARCH can fit the properties of stock market returns: dynamic and non-normal. Univariate AR(1)-GARCH(1, 1) model is used to study the marginal distribution of each index return, while copula is used to analyze the dependence between the two marginal distributions. Copula families offer various alternatives to the common assumption of normal dependence, including constant and time-varying. After fitting the marginal distributions into the constant and time-varying copula, we find constant t-copula and time-varying normal copula can explain the dependence between Shanghai and Shenzhen stock markets better. We also find evidence that there is obvious conditional dependence between the two stock markets.

KEYWORDS: stock return; dependence; copula; GARCH.

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1 Introduction

With the development of economics, the economists find there is weak or strong correlation between different economic markets. To analyze these relationships can be benefit to decide the portfolio and reduce the risk of investment. By discovering the changes of stock markets is a sign of changes in economic and with the deepening of financial integration, to study the dependence between different stock markets become very important.

It has been widely reported that the univariate distributions of the economic variables are non-normal but with excess kurtosis or fat tails, sometimes also exhibit skewness, and they are volatile over time, the variance of them are not constant. Because of this, the traditional models can not fit this kind of series. Since autoregressive conditional heteroskedasticity (ARCH) processes studied by Engle (1982) and generalized autoregressive conditional heteroskedasticity (GARCH) processes studied by Bollerslev (1986) appeared, it become possible to analyze economic time series. Engle and Bollerslev use Student's t distribution to solve the kurtosis or fat tails in the error distribution, this method is used by most of the authors. And then Nelson (1991) uses generalized exponential distribution, Hansen (1994) uses skewed t density function. In an extension, Jondau and Rockinger (2003a, b) add the skewness and kurtosis of the skewed Student's t distribution into consideration, and they show how to model data as a skewed Student's t distribution. They also show how to compute the cumulative distribution function and its inverse.

We use copula theory to analyze the dependence between daily stock returns of Shanghai Composite Index and Shenzhen Component Index over 6 years. We consider various copula families to fit our data, like normal copula, student's t copula, gumbel copula, clayton copula and frank copula. As a result, we provide evidence that for constant copula the student's t copula fits the returns best. We

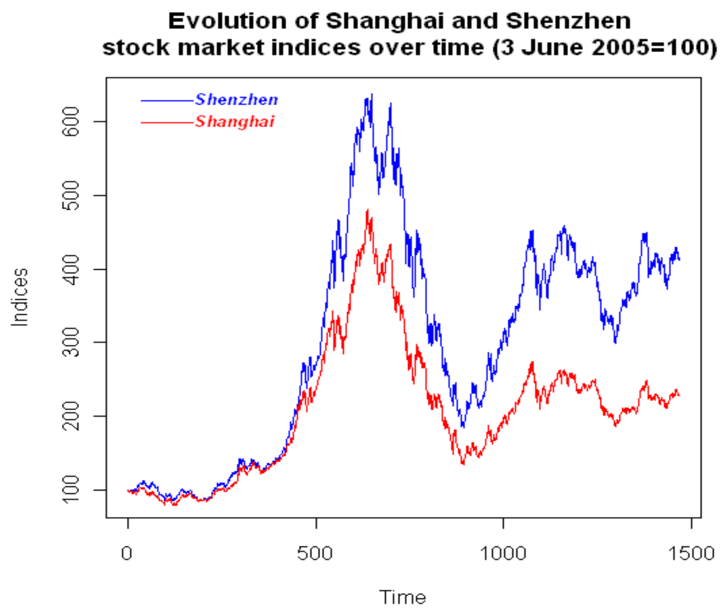

Figure 1: Evolution of Shanghai and Shenzhen stock market indices over time (3 June 2005=100).

also try time-varying normal copula, it is a main part of our thesis.

The remainder of this paper is as follows: In section 2, we analyze the data we used. In section 3, we present the theory of GARCH and copula. The estimation results are discussed in Section 4. Section 5 is the conclusion.

2 Data

This paper selects the Daily Closing Price of Shanghai Composite Index and Shenzhen Component Index from January 3rd, 2005 to March 18th, 2011, the sample size is 1468. The data source is the website of Yahoo Finance. We define the two daily closing prices on January 3rd, 2005 both equal to 100, and the transformed original data is shown in Figure 1.

Figure 2: Daily Index Return of Shenzhen and Shanghai Closing Prices.

2.1 The Index Return

This paper would like to analyze the Index Returns, as usual we use the log-difference of each daily closing prices:

$$R_t = 100 \times (log(p_t) - log(p_{t-1}))$$

where p_t means the closing price at the period of t. Figure 2 shows the two return series. As shown in this figure that neither returns had a significant trend or constant over our period.

The results of ADF test show that the two series are both stationary. Then we can use them to fit our models.

2.2 Summary Statistics

Table 1 presents some summary statistics of the data. As shown in this table, both of the means are close to zero and the standard deviations are very small, means neither of the two series has a constant term and all the data are around the mean.

The skewness and kurtosis tests show that both of the two series exhibit slight negative skewness, and excess kurtosis. The null hypothesis of Jarque-Bera test has been rejected under 0.05 significant level, which means neither of the series are unconditional normality distributions. Null hypothesis of ARCH-LM test has been rejected at the 0.05 level, thus both series own ARCH effects. The unconditional correlation coefficient between the two series indicates there is high linear dependence.

Table 1: **Summary Statistics.**

	Shanghai		Shenzhen
Mean	0.056634		0.096963
Std. Dev.	1.983267		2.197347
Skewness	-0.359236		-0.410347
Kurtosis	6.344923		5.826842
Jarque-Bera statistic	715.4503*		529.7324*
ARCH LM statistic	67.9362*		63.2258*
Linear correlation		0.934267	
Number of obs.		1467	

Note: This table shows some summary statistics of the 100 times log-differences of Shanghai Composite Index daily closing price and Shenzhen Component Index daily closing price. The sample period runs from January 3rd, 2005 to March 18th, 2011, yielding 1,468 observations in total. The ARCH-LM test of Engle (1982) is conducted using 16 lags. An asterisk (*) indicates a rejection of the null hypothesis at the 0.05 level.

3 Methodology

3.1 GARCH

3.1.1 The GARCH(p, q) process

The GARCH(p, q) (Generalized Autoregressive Conditional Heteroskedasticity) process introduced in Bollerslev (1986) is the extension of the ARCH process.

The GARCH(p, q) process is given by

$$\varepsilon_t|\psi_{t-1} \sim D(0, h_t), \tag{1}$$

$$\begin{aligned} h_t &= \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i h_{t-i}, \\ &= \alpha_0 + A(L)\varepsilon_t^2 + B(L)h_t, \end{aligned} \tag{2}$$

where

$$\begin{aligned} &p \geq 0,\ q > 0, \\ &\alpha_0 > 0,\ \alpha_i \geq 0,\ i = 1, \ldots, q, \\ &\beta_i \geq 0,\ i = 1, \ldots, p. \end{aligned}$$

And

$$\varepsilon_t = y_t - x_t' b \tag{3}$$

where ε_t is a real-valued discrete-time stochastic process, and ψ_t is the information set of all information through time t. y_t is the dependent variable, x_t is a vector of explanatory variables, and b is a vector of unknown parameters.

In the ARCH(q) process the conditional variance is defined as a linear function impacted by the past sample variances only. ARCH equations are shown as below:

$$\begin{aligned} y_t &= x_t\beta + \epsilon_t, \\ h_t &= \alpha_0 + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2, \\ \epsilon_t|\psi_{t-1} &\sim N(0, h_t), \end{aligned} \tag{4}$$

whereas in GARCH(p, q) process it is also impacted by lagged conditional variances. Thus, in Equation (2), for $B(L) = 0$, GARCH(p, q) process is simplified as the ARCH(q) process, and for $A(L) = B(L) = 0$, ε_t belongs to white noise.

In empirical, the most common used process is GARCH(1,1), which is given by Equation (1) and

$$h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 h_{t-1}, \tag{5}$$

where $\alpha_0 > 0$, $\alpha_1 \geq 0$, $\beta_1 \geq 0$.

3.1.2 The models for the GARCH

Let X_t and Y_t be two random variables, and $H(x, y)$ their joint probability distribution. Then the marginal distribution of X_t is just the distribution of X_t, with Y_t being ignored.

Here, we denote the log-difference of the Daily Closing Price of Shanghai Composite Index as the variable X_t and the log-difference of the Daily Closing Price of Shenzhen Component Index as the variable Y_t. The models for the marginal distribution are shown as below:

$$\begin{cases} X_t = \mu_x + \phi_{1x} X_{t-1} + \varepsilon_t \\ \sigma^2_{x,t} = \omega_x + \beta_x \sigma^2_{x,t-1} + \alpha_x \varepsilon^2_{t-1} \\ \varepsilon_t | \psi_{t-1} \sim D(0, \sigma^2_{xt}) \\ \varepsilon_t = z_t \sigma_{xt} \\ z_t \sim iid(0,1) \end{cases} \tag{6}$$

$$\begin{cases} Y_t = \mu_y + \phi_{1y} Y_{t-1} + \eta_t \\ \sigma^2_{y,t} = \omega_y + \beta_y \sigma^2_{y,t-1} + \alpha_y \eta^2_{t-1} \\ \eta_t | \psi_{t-1} \sim D(0, \sigma^2_{yt}) \\ \eta_t = z_t \sigma_{yt} \\ z_t \sim iid(0,1) \end{cases} \tag{7}$$

3.2 Copula

If there are uniform univariate marginal distribution functions in a multivariate distribution function, the multivariate distribution function is the Copula. Namely, a copula is a function which can link two or more marginal distributions together to form a joint distribution.

Since Sklar (1959) described the very beginning theorem of the relationship between a joint distribution and its marginal distributions, copula has been widely analyzed and applied in statistics. Such as Clayton (1978), Schweizer and Wolff (1981). From the beginning of 21th century or so, copula began to be used in economics and finance. See, for instance, the work of Cherubini et al. (2004), Goorbergh (2004), Berg et al. (2005), Koziol and Kunisch (2005), Jondeau and Rockinger (2006), Patton (2006) and Patton et al. (2009).

In this subsection, we first introduced the brief definition of bivariate copula, which is largely taken from Nelsen (2006). Next we define Sklar's theorem. Then we extend Sklar's theorem with conditioning information. After that we also brief state tail dependence. Finally we introduce constant normal copula and time-varying normal copula.

3.2.1 Definition of Copula

Definition 3.1 (Copula)**.** A two-dimensional Copula is a function C: $[u_1, u_2] \times [\nu_1, \nu_2] \to [0, 1]$, and satisfied those additions below:

(1) Boundary conditions

For every $u, \nu \in [0, 1]$,

$$1.C(0, \nu_1) = C(u_1, 0) = 0,$$
$$2.C(u_1, 1) = u_1, C(1, \nu_1) = \nu_1.$$

(2) Monotonic conditions

For every $u_1, \nu_1, u_2, \nu_2 \in [0, 1]$, when $u_1 \leq u_2, \nu_1 \leq \nu_2$,

$$3.C(u_2, \nu_2) + C(\nu_1, u_1) - C(u_2, \nu_1) - C(u_1, \nu_2) \geq 0.$$

Figure 3 shows the contour plots of several bivariate distributions which are all with standard normal marginal distributions and all the linear correlation coefficient ρ equals to 0.5.

3.2.2 Sklar's Theorem

Theorem 3.2 (Sklar's Theorem)**.** *Let* $H(x, y) = P[X \leq x, Y \leq y]$ *be a joint distribution function with margins* $F(x) = P(X \leq x)$ *and* $G(y) = P(Y \leq y)$.

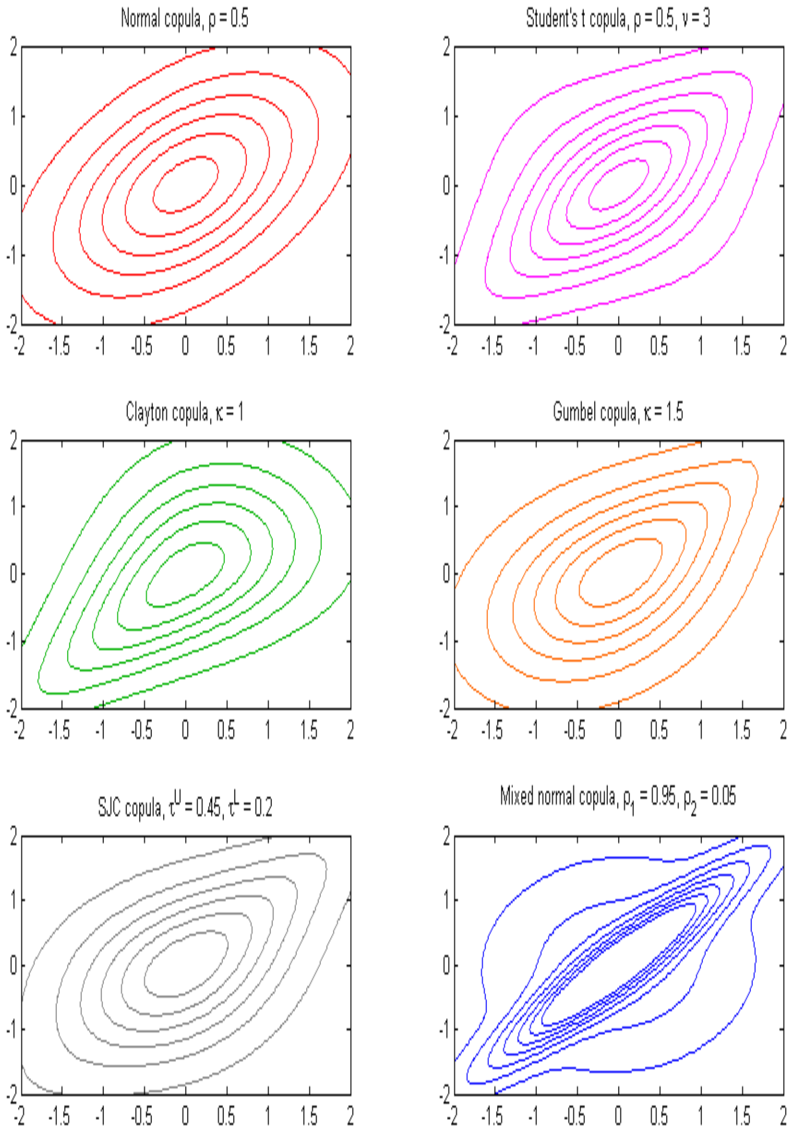

Figure 3: Contour Plots of Various Distributions All with Standard Normal Marginal Distributions and Linear Correlation Coefficients of 0.5.

Then there exists a copula $C : [0,1]^2 \rightarrow [0,1]$ *such that*

$$H(x,y) = C(F(x), G(y)) \tag{8}$$

if F and G are continuous, then C is unique; otherwise C is unique on $Ran(F(x)) \times Ran(G(y))$. *Namely, if C is a copula and* $F(x)$ *and* $G(y)$ *are distribution functions, then the function* $H(x,y)$ *defined by Equation (8) is a joint distribution function with margins* $F(x)$ *and* $G(y)$.

A proof can be seen in Nelsen (2006).

Sklar's theorem implies that a joint distribution can be factored into two marginal distributions of the components and a copula describes the dependence between the components. With Sklar's theorem, one can first estimate suitable marginal distributions of the components of a multivariate system by any possible method, then link them together through an appropriate copula to form a joint distribution. This is much easier than looking for multivariate extensions of univariate distributions as usual.

3.2.3 The Conditioning Information

Unlike the previous theorem that assumed the observations are iid, here we consider the conditioning information. With the conditioning information we can model the time variation of the joint process.

The copula theory is applicable to multivariate case, and in this article, we focus on bivariate distributions.

We define X_t and Y_t are two variables at time t, and the conditional variable is W_{t-1}, which contains all the information up to time $t-1$. Denote the conditional distribution of $X_t|W_{t-1}$ and $Y_t|W_{t-1}$ as $F_{X_t|W_{t-1}}$ and $F_{Y_t|W_{t-1}}$. Then Sklar's theorem can be extended to continuous conditional distributions; see Patton (2006).

That is the conditional distribution of $F_{(X_t,Y_t)|W_{t-1}}$ can be uniquely factored into the conditional marginal distributions of $F_{X_t|W_{t-1}}$ and $F_{Y_t|W_{t-1}}$, and a copula is conditional on W_{t-1}.

This extended theorem implies that any two univariate distributions, no matter whether they belong to the same family or not, can be linked together through any types of a copula.

The only complication is that using this theorem we must be sure all the conditional distributions are with the same information set.

3.2.4 Tail dependence

Tail dependence (Joe 1997) describes the dependence of the tails of the bivariate distribution between left lower quadrant and right upper quadrant. It is the Copula of the two variables.

Definition 3.3 (Tail dependence)**.** Let X and Y be two variables, F and G the distribution of X and Y. If the limit exists, the tail dependence of X and Y can be defined as:

$$\tau^U = \lim_{u \to 1} P(X > F^{-1}(u) | Y > G^{-1}(u)) \tag{9}$$

If $\tau^U \in (0, 1]$, the joint distribution of X and Y exhibits upper tail dependence; If $\tau^U = 0$, there is no upper tail dependence.

The lower tail dependence is similar to the upper tail dependence. If limit

$$\tau^L = \lim_{u \to 0} \frac{C(u, u)}{u}$$

exists, when $\tau^L \in (0, 1]$, copula C exhibits lower tail dependence; if $\tau^L = 0$, there is no lower tail dependence.

3.2.5 Normal Copula

The normal copula is the dependence function associated with bivariate normality, and is given by

$$C(u, v|\rho) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{(1-\rho^2)}} exp\left\{\frac{-(r^2 - 2\rho rs + s^2)}{2(1-\rho^2)}\right\} drds, -1 < \rho < 1 \tag{10}$$

where Φ^{-1} is the inverse of the standard normal $c.d.f.$ We define ρ_t as the value taken by the dependence parameter at time t, which is assumed to be driven by the following model:

$$\rho_t = \widetilde{\Lambda}\left(\omega_p + \beta_p \cdot \rho_{t-1} + \alpha \cdot \frac{1}{10}\sum_{j=1}^{10} \Phi^{-1}(u_{t-j}) \cdot \Phi^{-1}(v_{t-j})\right) \tag{11}$$

where $\widetilde{\Lambda} \equiv (1 - e^{-x})(1 + e^{-x})^{-1} = tanh(x/2)$ is the modified logistic transformation, in order to keep $\rho_t \in (-1, 1)$.

In Equation (11) we use an average of the last 10 lags rather than just the last lag, the last item in Equation (11) is the mean of the last 10 observations of the transformed variables $\Phi^{-1}(u)$ and $\Phi^{-1}(v)$, to capture any variation in dependence.

4 Results

4.1 Estimation of the GARCH

Before we estimate the copula model, we should estimate the two marginal distribution models separately. First, we fit Shanghai and Shenzhen index returns into models (6) and (7), and then use the results to get the probability integral transforms, U and V. The ordinary conditional distribution is always normal distribution. However, considering the financial data exist characters of fat-tail and skewness, we also take Student's t, Skewed Student's t distribution into consideration. We will call the above model under the three conditional distributions to estimate the two stock market marginal distributions, and compare results of the three distributions. The estimate results are presented in Table 2.

Table 2: **Results for the Shanghai and Shenzhen stock return marginal distribution.**

	Normal	Student's t	Skewed Student's t
Shanghai return X_t			
Intercept	0.121*	0.160 *	0.096*
Φ_{1x}	-0.008	-0.003	-0.006
ω_x	0.059*	0.042*	0.046*
β_x	0.923*	0.936*	0.935*
α_x	0.063*	0.056*	0.055*
Shape ν_x		4.823*	5.041*
Skewness λ_x			0.892*
Log likelihood	-2981.021	-2918.906	-2913.596
Shenzhen return Y_t			
Intercept	0.133*	0.170 *	0.110*
Φ_{1y}	0.030	0.021	0.019
ω_y	0.097*	0.066*	0.066*
β_y	0.920*	0.935*	0.934*
α_y	0.061*	0.054*	0.054*
Shape ν_y		5.179*	5.369*
Skewness λ_x			0.903*
Log likelihood	-3155.026	-3102.823	-3098.794

"*" significant level=0.05

In Table 2, except parameters Φ_{1x} and Φ_{1y}, all the parameters of the model are significant at the level 0.05 for both Shanghai index return and Shenzhen index return. Considering the maximum log-likelihood, we think the Skewed Student's t distribution fits best no matter for Shanghai or Shenzhen. Transform the results of Skewed Student's t distribution into probability integral U (Shanghai) and V

(Shenzhen), and do the Kolmogorov-Smirnov test for the density specification. The p-value of K-S test for Shanghai return is 0.4831, while p-value of Shenzhen return is 0.7048. Both of them show U and V are from Uniform (0, 1). In Figure 5 (Appendix D) we present the plot of probability integral transforms U and V. As shown in Figure 5, there is a close link between the tails of U and V.

4.2 Estimation of the Copula

Copula has various types and now we present our main results of this paper. Because the dependence of the two returns is dynamic, we chose time-varying copula and constant copula to describe the dependence separately. We estimate different copulas for the Shanghai and Shenzhen index returns and compare them to find which copula can explain the characters of relations of the two stock returns. The estimated results for various copulas are shown in Table 3. It includes constant copula of elliptical family copula, archimedean family, and time-varying copula of normal distribution. After estimating the results of various models, we plot the constant ρ and conditional correlation ρ_t for normal copula (Figure 3). We also present the contour curves of U and V (shown in top-left of Figure 6), and the other contour plots are estimated results for different copulas.

All the parameters of different copula shows the dependence between Shanghai and Shenzhen return is high. Considering the maximum log-likelihood, the constant Student's t copula and time-varying normal copula fit best. The tail dependence of Student's t copula is symmetric and equal to 0.684 which is bigger than 0.4. That means the tail dependence is very high. The result is the same as we show in Appendix D (Figure 5). On the other hand, time-varying normal copula also fits the data very well. Not only is its log-likelihood larger than constant normal copula, but it can show the dynamic conditional correlations. The comparison between correlation of constant normal copula and correlations of

Table 3: **Copula models results for Shanghai and Shenzhen return.**

	Copula	Log likelihood
Constant Copula		
Normal		
ρ	0.924*	1402.142
	(0.004)	
Student's t		
ρ	0.925*	1460.870
	(0.004)	
df	3.810*	
	(0.551)	
Gumbel		
τ^U	0.797*	1338.410
	(0.006)	
Clayton		
τ^L	0.852*	1264.158
	(0.006)	
Frank		
τ	0.749*	1289.113
	(0.019)	
Time-varying Copula		
Normal		
Ω	5.538*	1402.834
	(0.215)	
α	-0.129*	
	(0.065)	
β	-2.376*	
	(0.426)	

"*" significant level=0.05

time-varying normal copula is shown in Figure 4.

Comparing the constant correlations and time-varying correlations in Figure 4, the wave motion of the time-varying correlations are around the constant correlation up and down. Although the values of time-varying correlations have some fluctuations, they are still close to the value of constant correlation.

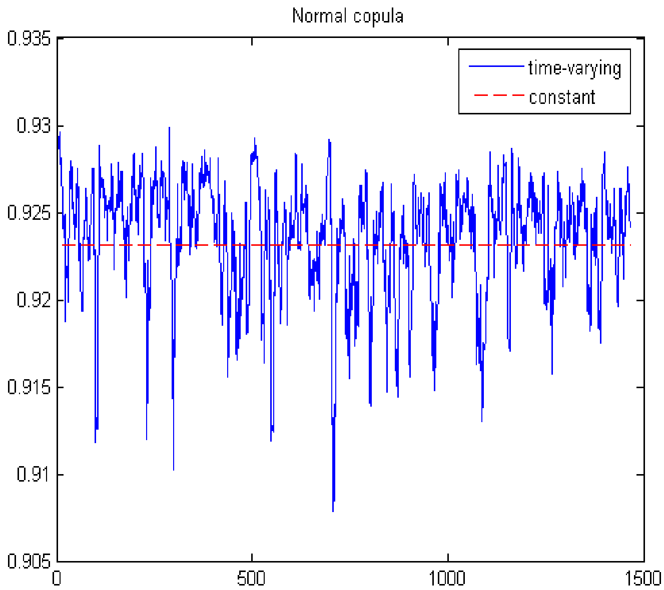

Figure 4: Conditional correlations in normal copula.

We also present the contour plot for estimated constant copulas in Appendix (Figure 6). From the contour plots, we can also see t-copula contour plot is the best constant copula model to fit the Shanghai and Shenzhen index return, which is the same as our conclusion in Table 3.

5 Conclusion

In the article, we concentrate on exploring the dependence between Shanghai index return and Shenzhen index return and use the conception of copula. We consider five constant copula models, including elliptical, archimedean family, and time-varying normal copula. First we find Skewed Student's t distribution of AR(1)-GARCH(1, 1) model separately explains the two returns best. Then we use maximum likelihood to estimate parameters of each copula models, and find constant Student's t copula and time-varying normal copula explain the dependence best. The constant t-copula gives the best description of dependence not only for the two stock returns, but also for the tail dependence. That means when the two returns go up or down together, the dependence will be larger. The time-varying normal copula also shows the dynamic conditional correlations. In the end, it is high probability of Shanghai index return and Shenzhen index return to be extreme values (no matter maximum or minimum) .

Copula can describe the dependence of multivariate distribution and ignore the limitation of marginal distribution. There are more and more types of copula, of which we can explore the other types for our case, such as symmetrized Joe-Clayton copula and time-varying symmetrized Joe-Clayton copula, Rotated Clayton copula. The different method to estimate the dependence of various copulas, for example, maximum pseudo-likelihood, IMF method (method of inference functions for margins) is also our future target.

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Appendices

A ARCH-LM test

The ARCH-LM test is a Lagrange multiplier(LM) test for autoregressive conditional heteroskedasticity(ARCH) in the residuals (Engle 1982), which can test wether there are ARCH effects. If the variance equation is correctly specified, there should be no ARCH left in the standardized residuals.This particular heteroskedasticity usually shows in observations of financial time series. The recent residuals appeared to be related to the past ones.

For the ARCH-LM test, the null hypothesis is: $H_0 : \beta_1 = \beta_2 = \ldots = \beta_q = 0$ and the alternative is: $H_1 : \beta_1 \neq 0$ or $\beta_2 \neq 0$ or $\ldots \beta_q \neq 0$. We run the regression:

$$\varepsilon_t^2 = \boldsymbol{\beta}_0 + \left(\sum_{i=1}^{q} \boldsymbol{\beta}_i \varepsilon_{t-i}^2 \right) + \boldsymbol{v}_t \tag{12}$$

where ε is the residual as we mentioned in GARCH equations, $\boldsymbol{v}_t$ assigns a spherical error process . This is a regression of the squared residuals on a constant item and lagged squared residuals up to order q.

Let K be the number of observations, the dimension of $\boldsymbol{\beta}_0$ is $\frac{1}{2}K(K+1)$ and for the coefficient matrices B_i with $i = 1, \ldots, q$, is $\frac{1}{2}K(K+1) \times \frac{1}{2}K(K+1)$.

This test statistic is χ^2 distributed with $(qK^2(K+1)^2/4)$ degree of freedom.

B Copula families

In this paper, we use these copulas below, where c presents the copula density and C is denoted as the cumulative distribution function. The more complete definitions can be seen in Goorbergh (2004).

B.1 Student's t

$$C_t(u, v; \rho, \nu) = \int_{-\infty}^{x_t} \int_{-\infty}^{y_t} p_t(s, t; \rho, \nu) ds dt$$

$$c_t(u, v; \rho, \nu) = \frac{1}{\sqrt{1-\rho^2}} \frac{\Gamma\left(\frac{\nu+2}{2}\right)\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)^2} \frac{\left[\left(1+\frac{x_t^2}{\nu}\right)\left(1+\frac{y_t^2}{\nu}\right)\right]^{-\frac{\nu+1}{2}}}{\left[1+\frac{x_t^2-2\rho x_t y_t+y_t^2}{\nu}\right]^{-\frac{\nu+2}{2}}},$$

where $x_t = t_\nu^{-1}(u)$, $y_t = t_\nu^{-1}(v)$, $\rho \in (0, 1)$, and $\nu > 0$. $t_\nu(\cdot)$ is the (univariate) Student's t distibution function; $p(\cdot, \cdot; \rho)$ denotes the bivariate Student's t density function with correlation coefficient ρ and degress of freedom ν:

$$p_t(x, y; \rho, \nu) = \frac{1}{\nu\pi\sqrt{1-\rho^2}} \frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\left(\frac{\nu}{2}\right)} \left[1 + \frac{x^2 - 2\rho xy + y^2}{\nu}\right]^{-\frac{\nu+2}{2}}.$$

Special cases are $C_t(u, v; -1, \nu) = W(u, v)$, $C_t(u, v; 0, \nu) = \Pi(u, n)$, and $C_t(u, v; 1, \nu) = M(u, n)$. Furthermore, we have $C_t(u, v; \rho, \infty) = C_N(u, v; \rho)$.

B.2 Clayton

$$C_C(u, v; \alpha) = (u^{-\alpha} + v^{-\alpha} - 1)^{-1/\alpha}$$

$$c_C(u, v; \alpha) = (1 + \alpha)(uv)^{-\alpha-1} C_C(u, v; \alpha)^{2\alpha+1},$$

where $\alpha \in [-1, \infty)/\{0\}$. Special cases include $C_C(u, v; -1) = W(u, v)$, $C_C(u, v; 0) = \Pi(u, v)$, and $C_C(u, v; \infty) = M(u, v)$.

B.3 Frank

$$C_F(u,v;\alpha) = \frac{1}{\alpha} log\left(1 + \frac{(e^{\alpha u}-1)\,(e^{\alpha v}-1)}{e^{\alpha}-1}\right)$$
$$c_F(u,v;\alpha) = \frac{\alpha}{e^{\alpha}-1}\frac{e^{\alpha(u+v)}}{\left(1+\frac{(e^{\alpha u}-1)(e^{\alpha v}-1)}{e^{\alpha}-1}\right)^2}$$

where $\alpha \in (-\infty, \infty)/\{0\}$. Special cases include $C_F(u,v;\infty) = W(u,v)$, $C_F;0 = \Pi(u,v)$, and $C_F(u,v;-\infty) = M(u,v)$.

B.4 Gumbel

$$C_G(u,v;\alpha) = exp\left\{-([-logu]^{\alpha} + [-logv]^{\alpha})^{1/\alpha}\right\}$$
$$c_G(u,v;\alpha) = \frac{(logu \times logv)^{\alpha-1}C_G(u,v;\alpha)}{uv([-logu]^{\alpha}+[-logv]^{\alpha})^{2-1/\alpha}}(\alpha - 1 - logC_G(u,v;\alpha)),$$

where $\alpha \in [1, \infty)$. Special cases are $C_G(u,v;1) = \Pi(u,v)$ and $C_G(u,v;\infty) = M(u,v)$.

The Clayton, Frank and Gumbel copula are belong to archmidean family copula. All except Clayton and Gumbel families are symmetric.

C Skewed Student's t density

The probability density function of Hansen's (1994) skewed t distribution is given by

$$g(z;\nu,\lambda)=\begin{cases} bc\left(1+\dfrac{1}{\nu-1}\left(\dfrac{bz+a}{1-\lambda}\right)^2\right)^{-\frac{\nu+1}{2}}, \text{ if } z<-a/b \\ bc\left(1+\dfrac{1}{\nu-2}\left(\dfrac{bz+a}{1+\lambda}\right)^2\right)^{-\frac{\nu+1}{2}}, \text{ if } z\geq -a/b \end{cases}$$

where the degrees of freedom parameter $\nu \in (2,\infty)$ and the skewness parameter $\lambda \in (-1,1)$. The constant a, b, and c are given by

$$\begin{aligned} a &= 4\lambda c\frac{\nu-2}{\nu-1}, \\ b^2 &= 1+3\lambda^2-a^2, \\ c &= \frac{1}{\sqrt{\pi(\nu-2)}}\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}. \end{aligned}$$

This density function has a zero mean and a unit variance. For $\lambda = 0$ one retrieves the Student's t density (with unit variance). As a consequence, the skewed t distribution specializes to the standard normal distribution for $\lambda = 0$ and $\nu = \infty$.

D Figure

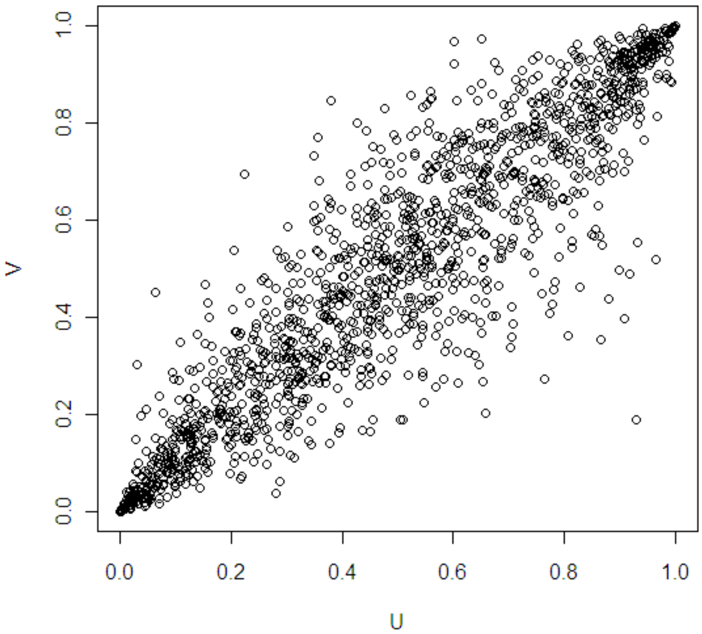

Figure 5: Plot of Shanghai and Shenzhen probability integral transforms.

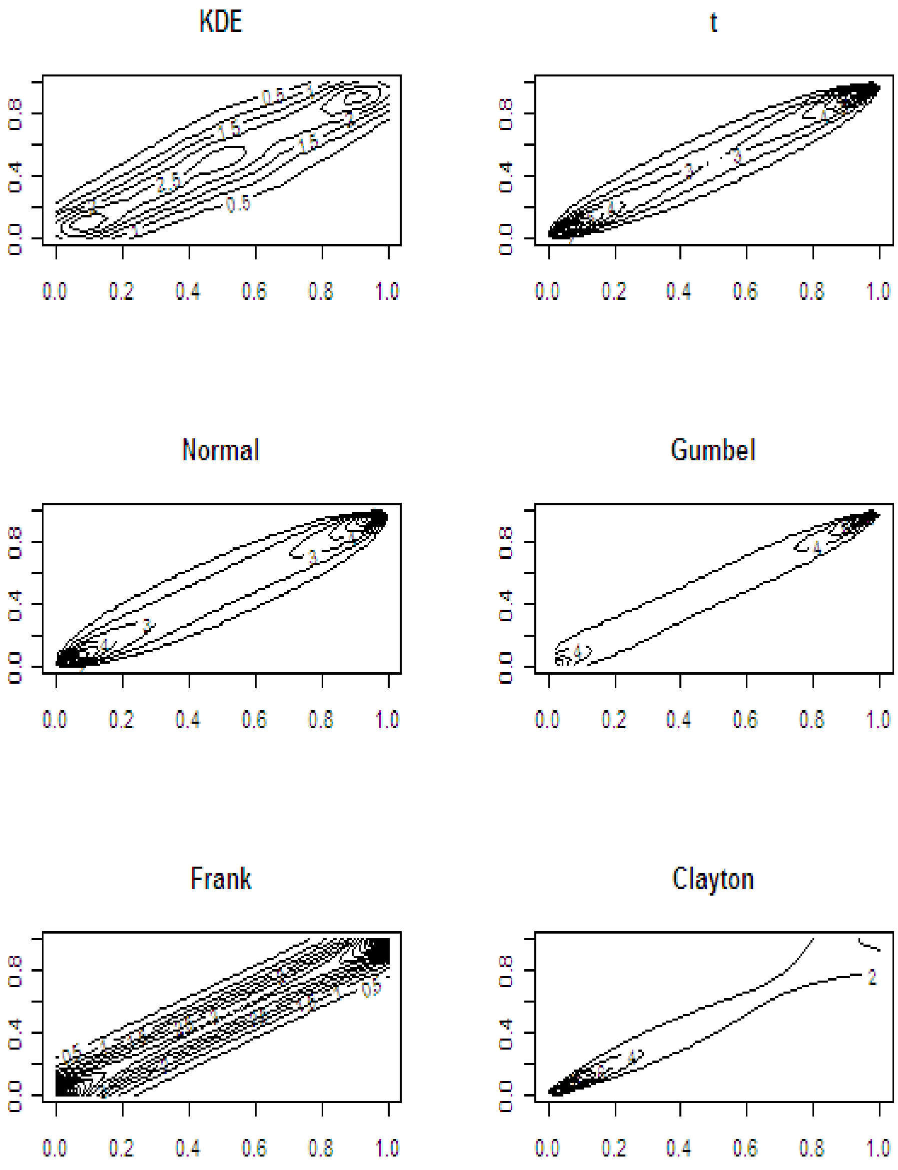

Figure 6: Contour plots for original data and various fitted copula.