Modeling and forecasting using U.S. Imports of Conventional Motor Gasoline data:an application of threshold autoregressive model

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Abstract

This paper concerns about specifying the threshold autoregressive model and forecasting. We consider theU.S. imports of conventional motor gasoline data and test threshold nonlinearityusing twomethods. One is Tsay's F test and the other one is Hansen's sup-LR test. Our data has threshold nonlinearity. Between two methods, forecasting from Hansen's method is better. Compared with linear autoregressive model, threshold autoregressive models generallyoutperform.

Key words: Threshold Autoregressive Model, NonlinearityTest,Tsay's F Test, Hansen's Sup-LR Test

1. Introduction

Nowadays, nonlinear time series models have been widely used in the economic field. Being affected by complex economic factors, time series data may have different behavior in different processes and may show nonlinearity. Nonlinear time series models open a new door for estimation and forecasting this kind of economic time series data. There is a typical nonlinear time series model - the threshold autoregressive (TAR) model- which is easy to do modeling with regime-switching data. In this paper, we are interested in the threshold autoregressive model and applyitto analyze the U.S. imports of conventional motor gasoline data.

The threshold autoregressive model describes complex dynamic data as an extension to autoregressivemodels. It is popular in application to nonlinear time series data.The TAR model is first introduced by Tong(1978). And then, Tong and Lim (1980)first complete the TAR model's exposition, and give effective technology for the practical issues in application (Geweke, 2007). And then, there are several scholars researched and developed test methods for the TAR model. For instance, Tsay (1989) proposed the F test which combined three studies of nonlinearity tests of Keenan (1985), Tsay (1986), and Petruccelli and Davies (1986). Hansen (1997) proposed the sup-LR test to study the threshold autoregressive.They are two popular approaches in recently studies.Scholars also have found a lot of good applications in different fields, such as economics, finance, ecology and public health. Particular in recently years, many scholars apply the TAR model to analyze real exchange rates, interest rates, and stock return etcin economic and finance field. (Johansson, 2001; Hardy, 2001; Hansen, 2011).

With the rapid development of modern industry, energy resource, such as crude oil, is consumed increasing faster than before. In the use of crude oil field, gasoline as a kind of products refined from crude oil has been widely used in the modern industry. One kind of gasoline – motor gasoline is widely used in our lives, since we need it to our cars or other vehicles. Therefore, motor gasoline price fluctuations will bring about living costs changes for people. This is the reason why more and more people pay more attention to the price of motor gasoline. The import of conventional motor gasoline is very important partfor the U.S. gasoline marketsupply. Therefore, it is the motivation for us to study the U.S.imports of conventional motor gasoline.

The structure of this paper is as follows. In Section 2, we will introducethe TAR model and two nonlinearity tests. In Section 3, we will describe theimport of conventional motor gasoline data. Section 4illustrates how to do modeling and forecasting using the TAR model. The last section is conclusion.

2. Method

2.1 The Threshold Autoregressive Model

The threshold autoregressive model is first proposed by Tong(1978). Tong and Lim (1980) first complete the TAR model's exposition. Based on theseliteratures,suppose a time series Y_t followsthethreshold autoregressive model below:

$$Y_t = \varphi_0^{(j)} + \sum_{i=1}^{p} \varphi_i^{(j)} Y_{t-i} + \varepsilon_t^{(j)}\,, \quad r_{j-1} \leq Y_{t-d} < r_j \qquad (1)$$

Where r_j(*j=1,…,k*) are the threshold values which belongto$-\infty = r_0 < r_1 < \cdots < r_k = \infty$; k is the number of regimes;$\varepsilon_t^{(j)} \sim \mathrm{iid}(0, \sigma^2)$. *d* is the threshold lag and *p*is the autoregressive order. Let's considerthe following model as an example including two regimes (k=2)and one threshold valuer_1.

$$Y_t = \begin{cases} \varphi_0^{(1)} + \sum_{i=1}^{p} \varphi_i^{(1)} Y_{t-i} + \varepsilon_t^{(1)}, \ \text{if } Y_{t-d} \leq r_1 \\ \varphi_0^{(2)} + \sum_{i=1}^{p} \varphi_i^{(2)} Y_{t-i} + \varepsilon_t^{(2)}, \ \text{if } Y_{t-d} > r_1 \end{cases} \qquad (2)$$

We will introduce test approaches based on model (2) in Section 2.

2.2 Tsay's Approach

This approach is first introduced by Tsay(1989)which proposed F statistic for the test in his paper. Tsay's F-testcan avoid knowing threshold values directly and makes the nonlinearity test more simply and widely utilized than before (Tsay, 1989; Zivot& Wang, 2005). The point of thisapproach is the use of the arranged autoregression with recursive least squares (RLS) estimation.

2.2.1 Testing Procedure

Observations need to be sorted according to the threshold values from the smallest observation to the largest observation.

Assume a set of observations $\left(Y_t, 1, Y_{t-1}, \dots, Y_{t-p}\right)$, where $t = p + 1, \dots, n$. For the threshold variable Y_{t-d}, there exist two situations. When $d \leq p + 1$, the threshold variables are$\left(Y_{p+1-d}, \dots, Y_{n-d}\right)$. On the other hand, when $d > p + 1$, the threshold variables are$(Y_1, \dots, Y_{n-d})$. Therefore, we combine two situations together: threshold variables$\{Y_h, \dots, Y_{n-d}\}$, where $h = \max\{1, p + 1 - d\}$. We sort them by a new time index π_i which expressnew order fromthe ith smallest observationin the set $\{Y_h, \dots, Y_{n-d}\}$. Therefore, $i = 1,2, \dots, n - d - h + 1$ and $n - d - h + 1$ is the effective sample size. Here, we use Y_{π_i} instead of Y_{t-d} to showthe threshold variable.For example, if the tenth observation in $\{Y_h, \dots, Y_{n-d}\}$ is the smallest,then $\pi_1 = 10 - d$.And then, the model (2) can be arranged as follow:

$$Y_{\pi_i+d} = \begin{cases} \varphi_0^{(1)} + \sum_{v=1}^{p} \varphi_v^{(1)} Y_{\pi_i+d-v} + \varepsilon_{\pi_i+d}^{(1)}, & \text{if } Y_{\pi_i} \leq r_1 \\ \varphi_0^{(2)} + \sum_{v=1}^{p} \varphi_v^{(2)} Y_{\pi_i+d-v} + \varepsilon_{\pi_i+d}^{(2)}, & \text{if } Y_{\pi_i} > r_1 \end{cases} \tag{3}$$

So the threshold variables Y_{π_i}which are smaller than $r_1$will fit to the first equation, while the threshold variables Y_{π_i}which are larger than r_1 will fit to the second equation.And then, Tsay usesrecursive least squares (RLS) estimates of φ_v in model

(3) to calculate the F statistic for testing the threshold nonlinearity.

Based on Ertel and Fowlkes (1976) and Goodwin and Payne (1977), the RLS estimates are calculated as follows:

$$\hat{\beta}_{m+1} = \hat{\beta}_m + K_{m+1}\left[Y_{d+\pi_{m+1}} - x'_{m+1}\hat{\beta}_m\right],$$

$$D_{m+1} = 1.0 + x'_{m+1}P_m x_{m+1},$$

$$K_{m+1} = P_m x_{m+1}/D_{m+1},$$

$$P_{m+1} = (I - P_m \frac{x_{m+1}x'_{m+1}}{D_{m+1}})P_m$$

Where $\hat{\beta}_m$is the vector of least squares estimates of model (3); $P_m$is the associated $X'X$ inverse matrix and $x_{m+1}$is the vector of regressors of the next observation to enter the autoregerssion$Y_{d+\pi_{m+1}}$.Then the predictive residual is

$$\hat{\varepsilon}_{d+\pi_{m+1}} = Y_{d+\pi_{m+1}} - x'_{m+1}\hat{\beta}_m \qquad (4)$$

and standardized predictive residual is

$$\hat{e}_{d+\pi_{m+1}} = \hat{\varepsilon}_{d+\pi_{m+1}}/\sqrt{D_{m+1}} \qquad (5)$$

(See Section 3.2 in Tsay, 1989)

And then, Tsay computes F statistic for testingthreshold nonlinearity as follows:

$$\hat{F}(p,d) = \frac{\left(\sum \hat{e}_t^{\,2} - \sum \hat{\epsilon}_t^{\,2}\right)/(p+1)}{\sum \hat{\epsilon}_t^{\,2}/(n-d-b-p-h)} \qquad (6)$$

where $\hat{\epsilon}_t$ is residual of regression below,

$$\hat{e}_{\pi_i+d} = \omega_0 + \sum_{v=1}^{p} \omega_v Y_{\pi_i+d-v} + \epsilon_{\pi_i+d} \qquad (7)$$

When there is existing threshold nonlinearity,$\hat{\omega}_v$ is statistically significant.This F statistic is an approximately F distributionwith degree of freedom $(p+1)$and$(n-d-b-p-h)$. Furthermore, $(p+1)\hat{F}(p,d)$is asymptotically a chi-squared random variable with degree of freedom $(p+1)$ (See Theorem 3.1 in Tsay, 1989)

2.2.2 Modeling Procedure

For modeling the TAR model, wefirstly select the order p via autocorrelation function (ACF) andpartial autocorrelation function(PACF). According to order p, we usually make the range of d, which is$d \leq p$. For all possible lags d, the number of possible (p, d) is p. And then, we can calculate the test statistic$\hat{F}(p, d)$p times. If reject the null hypothesis of linearity,it is possible to choose lag d when the maximum F statistic is obtained.That means we choose the lag d when the P-value of$\hat{F}(p, d)$is minimum.

After above steps, we already gain order p, lag d and the predictive residuals, and then we need to locate the threshold values. Tsay(1989) suggests usinga figure - "the scatter plot of the t-statistics of recursive least squares estimates versus the order threshold variable" to locate the threshold value.

BesidesTsay's approach, we will introduce another method.

2.3 Hansen's Approach

In Hansen (1997), model (2) is rewritten as:

$$Y_t = \left(\varphi_0^{(1)} + \sum_{i=1}^{p} \varphi_i^{(1)} Y_{t-i}\right) I\,(Y_{t-d} \leq r_1) + \left(\varphi_0^{(2)} + \sum_{i=1}^{p} \varphi_i^{(2)} Y_{t-i}\right) I(Y_{t-d} > r_1) + \varepsilon_t (8)$$

where $\varepsilon_t \sim \mathrm{iid}(0, \sigma^2)$. The advantage of Hansen's method is that the threshold values can be estimated together with other model parameters. However, it also has limitation that the method in Hansen (1997) is only able to apply in TAR with two regimes.

Hansen (1997) use Sup-LR test to test threshold nonlinearity. The likelihood ratio testis computed as follows:

$$F(r_1) = \frac{RSS_0 - RSS_1}{\hat{\sigma}_1^2(r_1)} = n' \frac{\hat{\sigma}_0^2 - \hat{\sigma}_1^2(r_1)}{\hat{\sigma}_1^2(r_1)} \quad (9)$$

Where $RSS_0$is the residual sum from the null hypothesis and $RSS_1$is the residual sum from the alternative hypothesis given the threshold valuer_1. $\hat{\sigma}_0^2$is the residual variance under the null hypothesis and $\hat{\sigma}_1^2$is the residual variance under the alternative hypothesis. This test is the standard F test. However, there is a problem thatis the threshold value is unknown.For solving this problem, Hansen (1997) proposed an approach that is sup-LR test to search all of possible values of the threshold variable. The equation of sup-LR is as follow:

$$F_s = \sup_{r_1 \in Y_d} F(r_1) \qquad (10)$$

where $Y_d$is the set of threshold variable. Then we choose the threshold value when $\hat{\sigma}_1^2(r_1)$ is minimum. The asymptotic distribution is non-standard, but the critical value is available (See Hansen, 1997, 2000).

3. Data

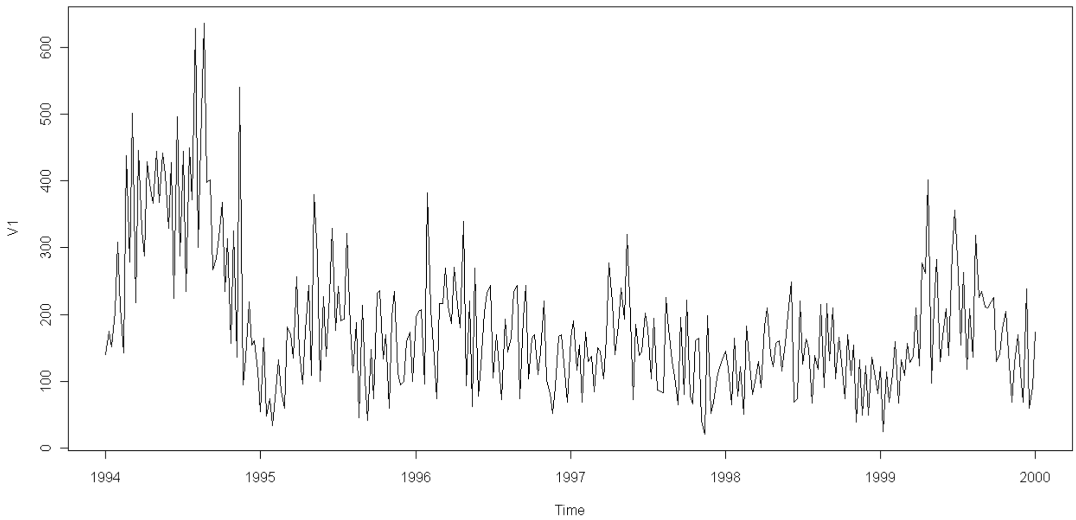

Figure 1: Weekly data of U.S. Imports of Conventional Motor Gasoline (Thousand Barrels per Day)

Our data is from U.S. Energy Information Administration (http://www.eia.doe.gov). It is Weekly U.S. Imports of Conventional Motor Gasoline (Thousand Barrels per Day), from January 07, 1994 to December 31, 1999.

4. Results

In this section, we will use U.S. imports of conventional motor gasoline data to do modeling and forecasting.

4.1 Modeling

4.1.1Choose Order p

First, ACF & PACF help us to determine order p.

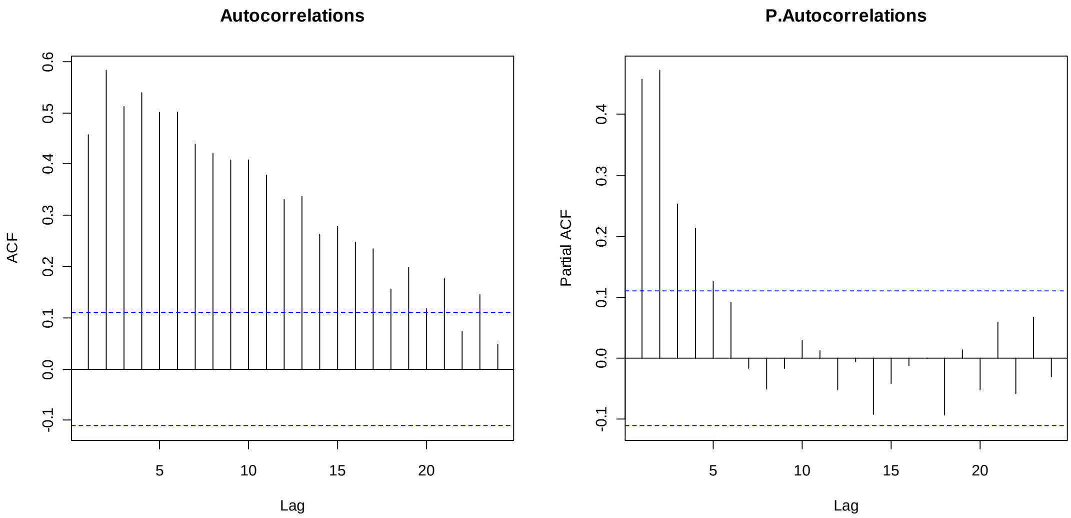

Figure 2: ACF & PACF

In Figure 2, ACF has a tail and the tail is cut off in PACF. We focus on PACF. After p=2, the PACF function is falling down rapidly, and after p=5, the significance of PACF is not strong. Since high order AR model can fit nonlinear dynamics well, we try lower order in nonlinear model (p=2). We also consider an AR (5) model to compare with nonlinear models.

4.1.2 Nonlinearity Test and Selecting the Delay Parameter d

Then we use Tsay's F test to test existence for threshold nonlinearity. The null hypothesis is no threshold nonlinearity.

Table 1: Nonlinearity test when p=2

	Threshold lags,d	
	1	2
F-value	3.6823	3.7813
P-value	0.0126	0.011

For both d=1 and d=2, the p-values are smaller than 0.05. We reject the null hypothesis that there is no threshold nonlinearity. Then we consider there exists threshold nonlinearity.

Generally speaking, we assume d is no more than p in model. For a given AR order p, Tsay suggests to select an estimate of the delay parameter, such that

$$d = \arg\max_{d \le p} \hat{F}(p, d_p)$$

Where $\hat{F}(p, d_p)$ is the F-statistic value. From Table 1, when d=1, F=3.6823 and when d=2, F=3.7813. So we take d=2.

Now we use Hansen's approach to test nonlinearity of the time series. We have just mentioned order p=2 and delay d=2. We also use these two choices to test nonlinearity. The result shows in the following table.

Table 2: Hansen sup-LR nonlinearity test

Number of Bootstrap Replications	1000
Threshold Estimate	154
F-test for no threshold	13.8753
Bootstrap P-Value	0.022

The null hypothesis is no threshold nonlinearity. Bootstrap P-Value=0.022. At the 0.05 significance level, we reject the null hypothesis. We also can get the threshold estimation and F-statistics from Hansen sup-LR nonlinearity test. In this series, threshold value r=154 and F=13.8753.

4.1.3 Locating Threshold Value

We identify the threshold value using scatter plot of t-statistic of the recursive least squares. The abscissa stands for ordered threshold variable and ordinate stands for t-statistics. In general case, when t-statistics is greater than 2, the result is significance. Then we check scatter plot.

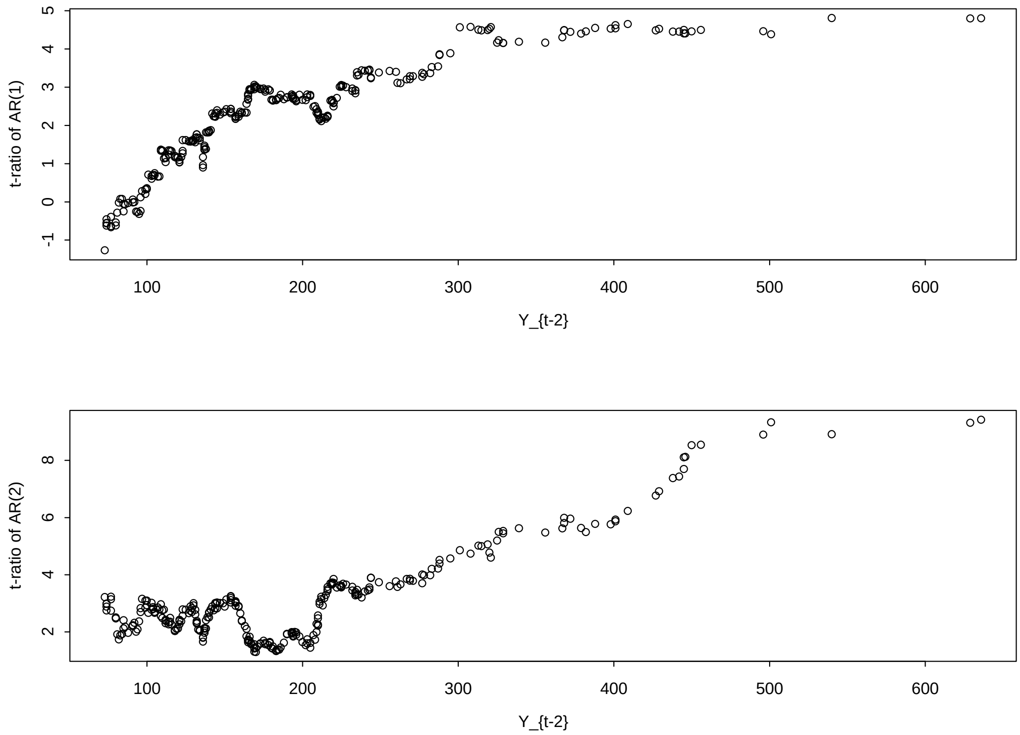

Figure 3: Scatter plot of Recursive t Ratios versus ordered threshold variable

From Figure 3, we identify threshold value r1=110 clearly, because there is an obvious jump around 110. The plot also shows there is a jump around 300. But after threshold value r2=300, there are only several observations. We may consider r2=300 is a possible threshold value.

In Table 2, we find threshold estimate is equal to 154 in Hansen's approach. We also use graphical tool to observe intuitively. From Figure 4, we can see that when threshold value is equal to 154, likelihood ratio statistics take the minimum.

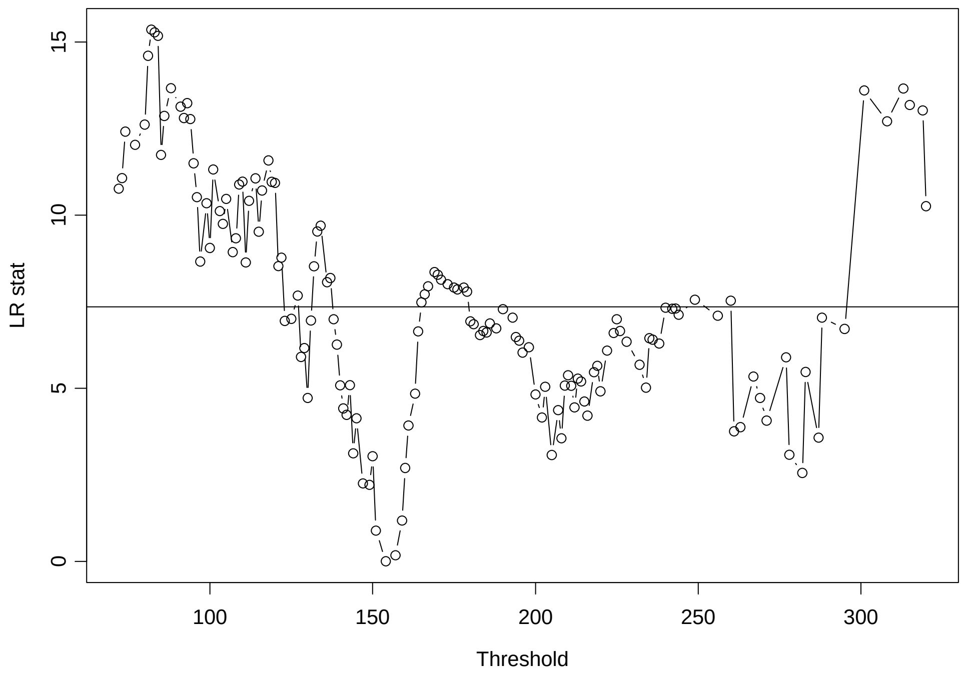

Figure 4: Confidence interval for threshold value by inverting LR statistics

4.1.4 Estimation

For both two methods, we obtain the following estimated models by using the results above.

Model1 is using by Tsay's approach with two-regime and the threshold value is r1=110.

$$\begin{aligned} Y_t &= 54.7642 + 0.1459Y_{t-1} + 0.8038Y_{t-2} \quad if \quad Y_{t-2} \le 110 \\ &= 33.6193 + 0.2368Y_{t-1} + 0.5401Y_{t-2} \quad if \quad Y_{t-2} > 110 \end{aligned}$$

Model2 is using by Tsay's approach with three-regime and two threshold values are r1=110 and r2=300.

$$\begin{aligned} Y_t &= 54.7642 + 0.1459Y_{t-1} + 0.8038Y_{t-2} \quad if \quad Y_{t-2} \le 110 \\ &= 67.7023 + 0.1964Y_{t-1} + 0.3786Y_{t-2} \quad if \quad 110 < Y_{t-2} \le 300 \\ &= 44.2069 + 0.3389Y_{t-1} + 0.4651Y_{t-2} \quad if \quad Y_{t-2} > 300 \end{aligned}$$

Model3 is using by Hansen's approach with two-regime and the threshold value is r1=154.

$$\begin{aligned} Y_t &= 71.782 + 0.171Y_{t-1} + 0.487Y_{t-2} \quad if \quad Y_{t-2} \le 154 \\ &= 1.457 + 0.253Y_{t-1} + 0.627Y_{t-2} \quad if \quad Y_{t-2} > 154 \end{aligned}$$

4.2 Forecasting

In Table 3, we predict 13 weeks ahead (from Jan 7th in 2000 to Mar 31st in 2000), from three threshold autoregressive models and one linear AR model.

Table 3: Forecasting

	True value	PV in model1	PV in model2	PV in model3	PV in AR(5)
1	240	152.3558	150.427	164.8533	128.9953
2	94	163.6271	178.0235	174.8856	142.158
3	121	166.7444	174.0653	173.8478	128.885
4	264	168.6178	173.9603	174.7583	144.7502
5	131	170.3795	172.4602	175.1773	145.4075
6	201	171.1976	171.8044	175.1899	144.7948
7	148	173.7288	172.8345	176.8814	146.8613
8	240	173.4279	171.2017	176.1222	148.4574
9	201	174.2187	171.5585	176.5669	151.1507
10	177	174.2557	170.807	176.146	152.0759
11	140	176.2515	172.7882	177.9761	153.5507
12	255	176.3511	172.0622	177.4289	154.972
13	34	176.4862	172.0956	177.7566	156.3892

We calculate root of mean squared errors and do a comparison in Table 4.

Table 4: Root of mean squared error

	Model1	Model2	Model3	AR(5)
RMSE	67.81162	68.97989	67.48328	72.82559

Model1 and Model2 are modeled by Tsay's approach. Compared with two models, root of mean squared error in Model1 is smaller than root of mean squared error in Model2, since we add a threshold value (r2=300) which may increase errors and risks.

When the model is more complex, the error may be greater. And after threshold value r2=300, there are only several observations. Model1 and Model3 have in common with their numbers of regimes. Model1 is estimated by Tsay's approach and Model3 is estimated by Hansen's approach. Comparing those two models, we found that Model3 is slightly better than Model1. The disadvantage in Hansen's approach is that we just can estimate one threshold value and consider two-regime TAR model. The root of mean squared error in linear AR(5) model is greatest, which obtains RMSE=72.82559. Compared with nonlinear models, the linear model is the worst. This shows TAR model is good to apply in nonlinear data.

5. Conclusion

This paper has applied Threshold Autoregressive Model in U.S. imports of conventional motor gasoline data. We have shown how to test nonlinearity, specify threshold value and forecast. We have used two methods (Tsay and Hansen) modeling and forecasting and compared them. Tsay's approach can select multiple threshold values. Hansen's approach just can choose one threshold value. We may consider how to expand to multiple threshold values using Hansen's approach. At last, we forecast 13 weeks ahead. The result shows Hansen's method is more accurate.

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